Groups, Fields, and Vector Spaces

Homework #1 (2010-2011)

The standard group operation is denoted by juxtaposition.

Q1: Elements of order 2

Suppose that $G$ has two group elements, $a$ and $b$, both of order 2, for which their group composition, $ab$, also has order 2. Show that $a$ and $b$ commute, namely, that $ab = ba$.

Q2. Normal subgroups

Definition: A subgroup $H$ of $G$ is said to be a “normal” subgroup if, for any element $g$ of $G$ and any element $h$ of $H$, the combination $ghg^{-1}$ is also a member of $H$.

A. Show that if $\varphi$ is a homomorphism from $G$ to some other group $R$, then the kernel of $\varphi$ is a normal subgroup of $G$. (We already showed that the kernel must be a subgroup, here we are to show that it is normal as well.)

B. Show that if $H$ is a normal subgroup and $b$ is any element of $G$, then the right coset $Hb$ is equal to the left coset, $bH$.

C. Show that if $H$ is a normal subgroup, then any element of the right coset $Hb$, composed with any element of the right coset $Hc$, is a member of the right coset $Hbc$, with the product $bc$ carried out according to the group operation in $G$.

D. Consider the mapping from group elements to cosets, $\varphi(b) = Hb$. Show that this constitutes a homomorphism from the group $G$ to the set of cosets, with the group operation on cosets defined by $(Hb) \circ (Hc) = Hbc$.

E. Find the kernel of the homomorphism in D.

Q3. Dihedral groups (one step beyond cyclic groups)

Consider the following distinct elements: $e$, $a$, and $r$. Assume that they compose in a way that obeys the associative law, that $e$ is the identity, that $a$ is of order 2, and that $r$ is of order $n \geq 2$. (Only $n \geq 3$ is interesting, though.) Suppose further that $a$ and $r$ satisfy $ra = ar^{n-1}$, and that the elements of the set $S = \{e, r, r^2, ..., r^{n-1}, a, ar, ar^2, ..., ar^{n-1}\}$ are all distinct. Show that this set constitutes a group, of size $2n$. (This is known as the “dihedral group” $D_n$.)