Groups, Fields, and Vector Spaces

Homework #2 (2010-2011)

Q1. Extensions of finite fields

Recall that $\mathbb{Z}_2$ is the field containing $\{0,1\}$, with addition and multiplication defined (mod 2). Consider the polynomial $x^4+x+1=0$. This has no solutions in $\mathbb{Z}_2$, so let's add a formal quantity ξ for which $\xi^4+\xi+1=0$ (and which satisfies the associative, commutative, and distributive laws for addition and multiplication with itself and with $\{0,1\}$), and see whether it generates a field.

A. Using $\xi^4+\xi+1=0$, express ξ^r in terms of 1, ξ, ξ^2, and ξ^3 for $r=1,...,15$.
B. Using part A, show that the powers of ξ generate a field of size 16. This is $GF(2,4)$.
C. Show that $\varphi(\xi)=\xi^2$ is an automorphism of $GF(2,4)$.

Q2. Intrinsic relationships among dual spaces, etc.

The point of this question is to further illustrate the distinction between vector spaces that have the same abstract structure just because they have the same number of dimensions, and vector spaces for which there is a natural, coordinate-free correspondence.

A. Find an intrinsic relationship (a.k.a. "canonical homomorphism") between V and V^{**}. (V^{**} is the dual of V^*, i.e., the space of mappings from elements φ of V^* to the field.) That is, find a linear mapping Φ from elements v of V to elements $\Phi(v)$ of V^{**}.

B. Find an intrinsic relationship (a.k.a. "canonical homomorphism") between $Hom(V,W)$ and $Hom(W^*,V^*)$. That is, find a linear mapping Z from elements φ of $Hom(V,W)$ to elements $Z(\varphi)$ of $Hom(W^*,V^*)$.

C. Find an intrinsic relationship (a.k.a. "canonical isomorphism") between $(V\otimes W)^*$ and $Hom(V,W^*)$. That is, (a) given an element B of $(V\otimes W)^*$, find a linear mapping Φ that takes elements B of $(V\otimes W)^*$ to elements $\Phi(B)$ of $Hom(V,W^*)$. (b) Given an element ξ of $Hom(V,W^*)$, find a linear mapping Ψ that takes elements ξ of $Hom(V,W^*)$ to elements $\Psi(\xi)$ of $(V\otimes W)^*$. (c) Show that Φ and Ψ are inverses, i.e., $\Psi(\Phi(B))=B$ and $\Phi(\Psi(\xi))=\xi$.

Q3. Parity

A. What is the parity of a cyclic permutation of q elements, i.e., the permutation that puts 2 where 1 was, puts 3 where 2 was, puts 4 where 3 was, …, puts q where $q-1$ was, and puts 1 where q was?

B. Recall the dihedral group: the symmetry group of a regular *n*-gon, containing rotations by $2\pi k / n$ radians, and reflections. (a) It can also be considered a permutation group, because it permutes the vertices of the *n*-gon. Which group elements correspond to a permutation with an even parity, and which to an odd parity? (b) The dihedral group can be considered a permutation group in another way, because it acts on the *edges* of the *n*-gon. In this representation, which group elements correspond to permutations with even parity, and which ones to an odd parity?