Linear Systems, Black Boxes, and Beyond

Homework #1 (2010-2011), Answers

**Q1: Fourier transforms, derivatives, and integrals**

Setup is \( \hat{s}(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt \), with \( s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{s}(\omega) e^{i\omega t} d\omega \).

A. For \( q(t) = \frac{d}{dt} s(t) \), find \( \hat{q}(\omega) \).

Using the “synthesis” integral,
\[
q(t) = \frac{d}{dt} s(t) = \frac{d}{dt} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{s}(\omega) e^{i\omega t} d\omega \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{s}(\omega) \frac{d}{dt} (e^{i\omega t}) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{s}(\omega) (i\omega e^{i\omega t}) d\omega .
\]

So the coefficient of \( e^{i\omega t} \) in \( q(t) = \frac{d}{dt} s(t) \) is \( \hat{q}(\omega) = i\omega \hat{s}(\omega) \).

B. For \( q_n(t) = \frac{d^n}{dt^n} s(t) \), find \( \hat{q}_n(\omega) \).

Iterating part A: \( \hat{q}_n(\omega) = i\omega \hat{q}_{n-1}(\omega) \), so \( \hat{q}_n(\omega) = (i\omega)^n \hat{s}(\omega) \).

C. For \( z(t) = \int_{-\infty}^{t} s(\tau) d\tau \), find \( \hat{z}(\omega) \).

Since \( s(t) = \frac{dz}{dt} \), we can use part A: \( \hat{s}(\omega) = i\omega \hat{z}(\omega) \), so, except possibly at \( \omega = 0 \), \( \hat{z}(\omega) = \frac{\hat{s}(\omega)}{i\omega} \).

D. Apply C to \( s(t) = \delta(t) \) to find a function whose Fourier transform, except possibly at 0, is \( \frac{1}{i\omega} \).

Since the Fourier transform of the delta-function is 1 everywhere, the integral of the delta-function, \( h(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \) has the required Fourier transform \( \frac{1}{i\omega} \). Since the delta-function is an infinitesimally narrow peak with a unit area, the integral evaluates as \( h(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \). This is the “Heaviside step function.” Its value at zero, which is formally undefined, is irrelevant for most purposes.
Q2: Fourier transforms and moments

Setup is $\hat{s}(\omega) = \int_{-\infty}^{\infty} s(t)e^{-i\omega t} dt$, with $s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{s}(\omega)e^{i\omega t} d\omega$, but now we are thinking of $s$ as a probability distribution.

A. Write the normalization condition $\int_{-\infty}^{\infty} s(t)dt = 1$ in terms of $\hat{s}(\omega)$.

Since $e^{i\omega t} = 1$ for $\omega = 0$, $\hat{s}(0) = \int_{-\infty}^{\infty} s(t)dt$, so the normalization condition is $\hat{s}(0) = 1$.

B. Write the mean (first moment) $\langle t \rangle = \int_{-\infty}^{\infty} ts(t)dt$ in terms of $s'(\omega) = \frac{d}{d\omega} \hat{s}(\omega)$.

Since $\hat{s}'(\omega) = \int_{-\infty}^{\infty} s(t) \frac{d}{d\omega} e^{-i\omega t} dt = \int_{-\infty}^{\infty} s(t)(-it)e^{-i\omega t} dt$, it follows that $\hat{s}'(0) = \int_{-\infty}^{\infty} s(t)(-it)dt$ and that $\int_{-\infty}^{\infty} ts(t)dt = is'(0)$.

C. Write the variance (second moment) $\langle (t - \langle t \rangle)^2 \rangle = \langle t^2 \rangle - \langle t \rangle^2 = \int_{-\infty}^{\infty} t^2 s(t)dt - \left(\int_{-\infty}^{\infty} ts(t)dt\right)^2$ in terms of $s'(\omega) = \frac{d}{d\omega} \hat{s}(\omega)$ and $s''(\omega) = \frac{d^2}{d\omega^2} \hat{s}(\omega)$.

As in part B, $\hat{s}''(\omega) = \int_{-\infty}^{\infty} s(t) \frac{d^2}{d\omega^2} e^{-i\omega t} dt = \int_{-\infty}^{\infty} s(t)(-t^2)e^{-i\omega t} dt$, so $\int_{-\infty}^{\infty} t^2 s(t)dt = -\hat{s}''(0)$.

So $\int_{-\infty}^{\infty} t^2 s(t)dt - \left(\int_{-\infty}^{\infty} ts(t)dt\right)^2 = -\hat{s}''(0) - (is'(0))^2 = -\hat{s}''(0) + (\hat{s}'(0))^2$.

Q3: The half-infinite cable (repeating indefinitely to the right)
This is to be viewed as a network of resistors and capacitors. Calculate the impedance of the system (input applied across terminals at left) in terms of the impedances $F(\omega)$, $G_1(\omega)$, and $G_2(\omega)$ for $F$, $G_1$, and $G_2$.

Hint: Let the composite system be $H$. Note the following, and then write an equation for $H(\omega)$.

The impedance of the composite system on the left is a series combination of three components: $G_1$, the parallel combination of $F$ and $H$, and $G_2$. Therefore its impedance is

$$G_1(\omega) + \frac{F(\omega)H(\omega)}{F(\omega) + H(\omega)} + G_2(\omega).$$

Since (as the hint indicates) this is equivalent to the entire half-infinite cable, $H(\omega) = G_1(\omega) + \frac{F(\omega)H(\omega)}{F(\omega) + H(\omega)} + G_2(\omega)$. Solving for $H(\omega)$ yields

$$H(\omega)^2 - G(\omega)H(\omega) - G(\omega)F(\omega) = 0,$$

where $G(\omega) = G_1(\omega) + G_2(\omega)$, or,

$$H(\omega) = \frac{G(\omega) + \sqrt{G(\omega)^2 + 4F(\omega)G(\omega)}}{2}.$$

Note concerning the continuum limit: This corresponds to allowing each subunit to represent progressively less and less length. Then $F$ has units of impedance/cm (and increases as the segment shortens), and $G$ has units of impedance-cm (and decreases as the segment shortens). In this limit, $H(\omega) \approx \sqrt{F(\omega)G(\omega)}$. This enables one to calculate the “cable length” $\lambda$, which is the distance required for the transmembrane current to fall by a factor of $e$. To do this, note that total transmembrane current $I_{\text{total}}$ is $\int_0^{\infty} e^{-x/\lambda} dx = \lambda$ times the current per unit length $I_{\text{peak}}$ at the injection site, but also, $I_{\text{total}} / I_{\text{peak}}$ is inversely proportional to the total cable impedance $H(\omega)$, divided by the impedance per unit length, $F(\omega)$. So $\lambda = \frac{H(\omega)}{F(\omega)} = \frac{\sqrt{G(\omega)}}{F(\omega)}$.

Q4. Boxcar smoothing

Boxcar smoothing refers to convolution with the function $s(t)$, where $s(t) = \begin{cases} \frac{1}{L} & |t| \leq L/2 \\ 0 & |t| > L/2 \end{cases}$. Find its Fourier transform. What does it look like? Is this a good way to smoothe?
\[
\hat{s}(\omega) = \int_{-\infty}^{\infty} s(t)e^{-i\omega t} \, dt = \frac{1}{L} \int_{-L/2}^{L/2} e^{-i\omega t} \, dt = \frac{1}{-i\omega L} \left[ e^{i\omega L/2} - e^{-i\omega L/2} \right] = \frac{\sin(\omega L/2)}{(\omega L/2)}.
\]

This (the “sinc” function) has a peak of 1 at \( \omega = 0 \), and descends in an envelope proportional to \( 1/|\omega| \) away from zero. There are zeros at \( \omega = (2\pi k / L) \), for \( k \neq 0 \). The center lobe (at \( \omega = 0 \)) is positive, but the adjacent lobes \( (\frac{2\pi}{L} < |\omega| < \frac{4\pi}{L}) \) are negative. So one problem with using this as a smoothing function is that it inverts the phase of non-negligible frequency components.

```matlab
>> x=[-8:0.01:8];
>> y=sinc(pi*x);
>> plot(x,y)
>> hold on;
>> plot([-8 8],[0 0],'k')
>> plot([0 0],[-0.5 1],'k')
>> set(gca,'YLim',[-0.2 1])
>> set(gca,'YLim',[-0.25 1])
>> xlabel('omega, as a multiple of 2pi/L')
>> set(gca,'XTick',[-8:8])
```