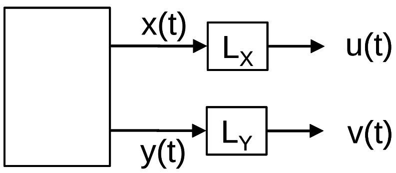
Linear Systems, Black Boxes, and Beyond

Homework #2 (2010-2011), Answers

Here, rectangles with letters inside represent linear filters, and rectangles with a "+" inside represents summation.

Q1: The reason for the normalization in the definition of coherency:



Given x and y, signals whose spectra $P_X(\omega)$ and $P_Y(\omega)$ cross-spectrum $P_{X,Y}(\omega)$, and coherency $C_{X,Y}(\omega)$ are known, find the spectra of u and v, their coherency, and coherence.

In terms of Fourier estimates (omitting the start-time argument), $F(u,\omega,T)=\hat{L}_X(\omega)F(x,\omega,T)$ and $F(v,\omega,T)=\hat{L}_Y(\omega)F(y,\omega,T)$, so

$$P_U(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|F(u,\omega,T)\right|^2\right\rangle=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|\hat{L}_X(\omega)F(x,\omega,T)\right|^2\right\rangle=\left|\hat{L}_X(\omega)\right|^2P_X(\omega),$$

$$P_V(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|F(v,\omega,T)\right|^2\right\rangle=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|\hat{L}_Y(\omega)F(y,\omega,T)\right|^2\right\rangle=\left|\hat{L}_Y(\omega)\right|^2P_Y(\omega),$$ and

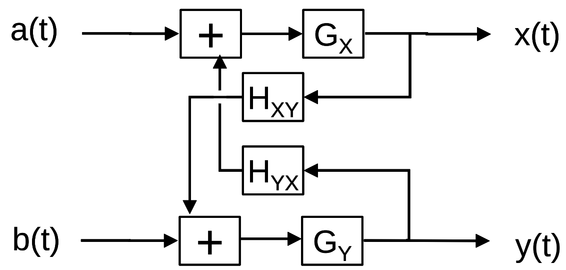
$$P_{U,V}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle F(u,\omega,T)\overline{F(v,\omega,T)}\right\rangle=$$
$$\lim_{T\to\infty}\frac{1}{T}\left\langle \hat{L}_X(\omega)F(x,\omega,T)\overline{\hat{L}_Y(\omega)}\overline{F(y,\omega,T)}\right\rangle=\hat{L}_X(\omega)\overline{\hat{L}_Y(\omega)}P_{X,Y}(\omega).$$

For the coherency,

$$C_{U,V}(\omega)=\frac{P_{U,V}(\omega)}{\sqrt{P_U(\omega)P_V(\omega)}}=\frac{\hat{L}_X(\omega)\overline{\hat{L}_Y(\omega)}P_{X,Y}(\omega)}{\left|\hat{L}_X(\omega)\right|\left|\hat{L}_Y(\omega)\right|\sqrt{P_X(\omega)P_Y(\omega)}}=\frac{\hat{L}_X(\omega)\overline{\hat{L}_Y(\omega)}}{\left|\hat{L}_X(\omega)\right|\left|\hat{L}_Y(\omega)\right|}C_{X,Y}(\omega).$$

Note that the multiplier $\dfrac{\hat{L}_X(\omega)\overline{\hat{L}_Y(\omega)}}{\left|\hat{L}_X(\omega)\right|\left|\hat{L}_Y(\omega)\right|}$ has magnitude 1, so $\left|C_{U,V}(\omega)\right|=\left|C_{X,Y}(\omega)\right|$ -- i.e., the coherence is unchanged by the interposed filters L_X and L_Y. These filters only affect the phase of the coherency, and this effect is manifest in a product of two terms – one that adds the phase shift due to L_X, and that subtracts the phase shift due to L_Y. In particular, if $L_X = L_Y$ (e.g., they are both due to the same measuring device), then these filters do not affect the coherency.

Q2. The cross-spectrum without explicit "common sources"



Given this setup, with A and B independent sources, find $P_X(\omega)$, $P_Y(\omega)$, and $P_{X,Y}(\omega)$ in terms of the power spectra of A and B.

In terms of Fourier estimates (omitting the argument T_0)

$$F(x,\omega,T) = \hat{G}_X(\omega)\left(\hat{H}_{YX}(\omega)F(y,\omega,T) + F(a,\omega,T)\right)$$

and

$$F(y,\omega,T) = \hat{G}_Y(\omega)\left(\hat{H}_{YX}(\omega)F(x,\omega,T) + F(b,\omega,T)\right)$$, so we can solve for $F(x,\omega,T)$:

$$F(x,\omega,T) = \hat{G}_X(\omega)\left(\hat{H}_{YX}(\omega)\hat{G}_Y(\omega)\left(\hat{H}_{YX}(\omega)F(x,\omega,T) + F(b,\omega,T)\right) + F(a,\omega,T)\right).$$

Writing $\hat{Z}(\omega) = \hat{G}_X(\omega)\hat{G}_Y(\omega)\hat{H}_{XY}(\omega)\hat{H}_{YX}(\omega)$ (corresponding to one round trip through the network, from a through G_X through H_{XY} through G_Y through H_{YX}),

$$F(x,\omega,T) = \frac{\hat{G}_X(\omega)\left(\hat{H}_{YX}(\omega)\hat{G}_Y(\omega)F(b,\omega,T) + F(a,\omega,T)\right)}{1-\hat{Z}(\omega)}$$, and similarly for $F(y,\omega,T)$,

$$F(y,\omega,T) = \frac{\hat{G}_Y(\omega)\left(\hat{H}_{XY}(\omega)\hat{G}_X(\omega)F(a,\omega,T) + F(b,\omega,T)\right)}{1-\hat{Z}(\omega)}.$$

Remembering that a and b are independent allows us to calculate the needed averages from

$P_X(\omega) = \lim_{T\to\infty}\frac{1}{T}\left\langle \left|F(x,\omega,T)\right|^2\right\rangle$, $P_Y(\omega) = \lim_{T\to\infty}\frac{1}{T}\left\langle \left|F(y,\omega,T)\right|^2\right\rangle$, and

$P_{X,Y}(\omega) = \lim_{T\to\infty}\frac{1}{T}\left\langle F(x,\omega,T)\overline{F(y,\omega,T)}\right\rangle$:

$$P_X(\omega) = \frac{\left|\hat{G}_X(\omega)\right|^2 P_A(\omega) + \left|\hat{G}_X(\omega)\hat{G}_Y(\omega)\hat{H}_{YX}(\omega)\right|^2 P_B(\omega)}{\left|1-\hat{Z}(\omega)\right|^2},$$

$$P_Y(\omega) = \frac{\left|\hat{G}_Y(\omega)\right|^2 P_B(\omega) + \left|\hat{G}_X(\omega)\hat{G}_Y(\omega)\hat{H}_{XY}(\omega)\right|^2 P_A(\omega)}{\left|1-\hat{Z}(\omega)\right|^2},$$

and

$$P_{X,Y}(\omega) = \frac{\left|\hat{G}_Y(\omega)\right|^2 \hat{G}_X(\omega)\hat{H}_{YX}(\omega)P_B(\omega) + \left|\hat{G}_X(\omega)\right|^2 \overline{\hat{G}_Y(\omega)\hat{H}_{XY}(\omega)}P_A(\omega)}{\left|1-\hat{Z}(\omega)\right|^2}.$$

Note that the denominator becomes unity if either cross-connecting filter *H* is absent.