Linear Transformations and Group Representations

Homework #1 (2010-2011), Answers

Q1: Eigenvectors of some linear operators in matrix form (also see Homework from "Algebraic Overview" (2008-2009))

In each case, find the eigenvalues, the eigenvectors, the dimensions of the eigenspaces, and whether a basis can be chosen from the eigenvectors.

A. $A=\begin{pmatrix}1 & r\\ 0 & 1\end{pmatrix}$.

First, use the determinant to find the eigenvalues. $\det(zI-A)=\det\begin{pmatrix}z-1 & -r\\ 0 & z-1\end{pmatrix}=(z-1)^2$.

$\det(zI-A)=0$ requires $z=1$, so the only eigenvalue of *A* is 1.

Say *V* has basis elements e_1 and e_2, expressed as columns $e_1=\begin{pmatrix}1\\0\end{pmatrix}$ and $e_2=\begin{pmatrix}0\\1\end{pmatrix}$. Then $Ae_1=e_1$ and $Ae_2=re_1+e_2$. So e_1 is an eigenvector of eigenvalue 1. To look for any others: Let $v=ae_1+be_2$. Then $Av=A(ae_1+be_2)=ae_1+b(re_1+e_2)=(a+br)e_1+be_2$.

$Av=v$ implies $ae_1+be_2=(a+br)e_1+be_2$. Since e_1 and e_2 are linearly independent (they form a basis), their coefficients must be equal. For e_1, this requires $a=a+br$, i.e., $b=0$. For e_2, the coefficients are always equal. So the only eigenvalues have $b=0$, i.e., the only eigenvalues are e_1 and its multiples.

So there is one eigenvalue, 1, whose eigenspace has dimension 1, spanned by the eigenvector $e_1=\begin{pmatrix}1\\0\end{pmatrix}$. Since *A* operates in a two-dimensional vector space, the eigenvectors cannot form a basis.

B. $B=\begin{pmatrix}q & r\\ r & q\end{pmatrix}$ *(assume* $q>r>0$*).*

Again, first use the determinant to find the eigenvalues.

$\det(zI-B)=\det\begin{pmatrix}z-q & -r\\ -r & z-q\end{pmatrix}=(z-q)^2-r^2$. $\det(zI-B)=0$ solves for $z=q\pm r$, so these are the eigenvalues of *B*. To find the eigenvectors: As in part *A*, say $e_1=\begin{pmatrix}1\\0\end{pmatrix}$ and $e_2=\begin{pmatrix}0\\1\end{pmatrix}$ and *v* is an eigenvector with $v=ae_1+be_2$. $Be_1=qe_1+re_2$. $Be_2=re_1+qe_2$. So

$Bv=aBe_1+bBe_2=a(qe_1+re_2)+b(re_1+qe_2)=(aq+br)e_1+(ar+bq)e_2$.

Looking for the eigenvector of eigenvalue $q+r$:

$Bv=(q+r)v$ implies $(aq+br)e_1+(ar+bq)e_2=(q+r)ae_1+(q+r)be_2$. Since e_1 and e_2 are linearly independent, equality can only hold if coefficients of e_1 match, and coefficients of e_2 match.
For e_1: $aq+br=aq+ar$; For e_2: $ar+bq=bq+br$. Both solve for $a=b$. So the eigenvectors corresponding to the eigenvalue $q+r$ are multiples of e_1+e_2, i.e., of $\begin{pmatrix}1\\1\end{pmatrix}$.

For the eigenvectors of eigenvalue $q-r$:

$Bv=(q-r)v$ implies $(aq+br)e_1+(ar+bq)e_2=(q-r)ae_1+(q-r)be_2$. Since e_1 and e_2 are linearly independent, equality can only hold if coefficients of e_1 match, and coefficients of e_2 match.
For e_1: $aq+br=aq-ar$; For e_2: $ar+bq=bq-br$. Both solve for $a=-b$. So the eigenvectors corresponding to the eigenvalue $q-r$ are multiples of e_1-e_2, i.e., of $\begin{pmatrix}1\\-1\end{pmatrix}$.
So there are two eigenvalues, $q+r$ and $q-r$, each with eigenspace of dimension 1, spanned by $e_1+e_2=\begin{pmatrix}1\\1\end{pmatrix}$, and $e_1-e_2=\begin{pmatrix}1\\-1\end{pmatrix}$. They form a basis.

Q2: Eigenvectors of some linear operators in a continuous space

V is a vector space of functions of time. In each case, find the eigenvalues and eigenvectors of the indicated operator, and determine whether the operator is time-translation invariant

A. $Lv(t)=tv(t)$.

If $v(t)$ has eigenvalue λ, then $Lv(t)=\lambda v(t)$ means $\lambda v(t)=tv(t)$, which means that either $t=\lambda$ or $v(t)=0$. This is satisfied by $v(t)=\begin{cases}a, t=\lambda\\0, t\neq\lambda\end{cases}$.

L is not time-translation invariant: $\left(D_T Lv\right)(t)=\left(D_T(tv)\right)(t)]=(t+T)v(t+T)$ but $\left(LD_T\right)(t)=L\left(D_T(v)\right)(t)=tv(t+T)$.

B. $Rv(t)=v(-t)$.

If $v(t)$ has eigenvalue λ, then $Rv(t)=\lambda v(t)$ means $\lambda v(t)=v(-t)$, and, that $\lambda^2 v(t)=\lambda v(-t)=v(t)$ which means that $\lambda=1$ or $\lambda=-1$.

If $\lambda=1$: Any even-symmetric function (satisfying $v(t)=v(-t)$) is an eigenvector.
If $\lambda=-1$: Any odd-symmetric function (satisfying $v(t)=-v(-t)$) is an eigenvector.

Starting with an arbitrary *f*, $(I+R)f = f + Rf$ is always an even-symmetric function, since $R(I+R)f = (R+R^2)f = (R+I)f$.

Similarly, $(I-R)f = f - Rf$ is always an odd-symmetric function, since $R(I-R)f = (R-R^2)f = (R-I)f = -(I-R)f$.

R is not time-translation invariant: $\left(D_T Rv\right)(t) = \left(D_T(Rv)\right)(t)] = v(-(t+T)) = v(-t-T)$ but $\left(RD_T\right)(t) = R\left(D_T(v)\right)(t) = v(-t+T)$.

Note that *R* and all the D_T 's form a group, since $R^2 = I$ and $D_T R = RD_{-T}$, vaguely like the dihedral group – but here, continuous and open-ended.

C. $Mv(t) = \frac{d}{dt}v(t)$.

If $v(t)$ has eigenvalue λ, then $Mv(t) = \lambda v(t)$ means $\lambda v(t) = \frac{dv}{dt}(t)$, i.e., that *v* satisfies the differential equation $\frac{dv}{dt} = \lambda v$. Solve by separation of variables

$\frac{dv}{v} = \lambda dt$, which implies $d(\log v) = \lambda dt$, which implies $v(t) = a\exp(\lambda t)$ (or by inspection). Eigenvectors are thus $v(t) = a\exp(\lambda t)$.

M is time-translation invariant: $\left(D_T Mv\right)(t) = \frac{d}{dt}v(t+T) = \left(MD_T v\right)(t)$.

D. $Nv(t) = t\frac{d}{dt}v(t)$.

If $v(t)$ has eigenvalue λ, then $Nv(t) = \lambda v(t)$ means $\lambda v(t) = t\frac{dv}{dt}(t)$, i.e., that *v* satisfies the differential equation $t\frac{dv}{dt} = \lambda v$.

Solve by separation of variables:

$\frac{dv}{v} = \lambda\frac{dt}{t}$, which implies $d(\log v) = \lambda d(\log t)$, which implies $v(t) = at^{\lambda}$ (or by inspection). Eigenvectors are thus $v(t) = at^{\lambda}$.

N is not time-translation invariant: $\left(D_T Nv\right)(t) = (t+T)\frac{d}{dt}v(t+T)$, but

$\left(ND_T v\right)(t) = t\frac{d}{dt}v(t+T)$.