Linear Transformations and Group Representations

Homework #2 (2010-2011), Answers

Q1: Classifying some operators

In each case, determine whether the operators are normal, self-adjoint, unitary, or projections, using the standard inner product for a finite-dimensional space (A, B, C, D), or for complex-valued functions on the line (E, F, G, H). Note: G and H are a bit harder.

A. $A=\begin{pmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$.

$A^*=\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$. $AA^*=\begin{pmatrix}3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ but $A^*A=\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$. So A is not normal (or self-adjoint, or unitary, or a projection). But $A^2=A$, so A is idempotent.

B. $B=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\varphi\\ 0 & -\sin\varphi & \cos\varphi\end{pmatrix}$.

This is a rotation by angle φ in the e_2-e_3-plane. $B^*=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\phi & -\sin\varphi\\ 0 & \sin\varphi & \cos\varphi\end{pmatrix}$. $BB^*=B^*B=I$, so B is normal and unitary (but not self-adjoint or a projection).

C. $C=\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\end{pmatrix}$.

This is like image projection: the first two coordinates are averaged, and the last two coordinates are averaged. $C^*=C$. So C is normal and self-adjoint. $C^2=C$, so C is a projection. $CC^*=C^2=C\neq I$, so C is not unitary.

D. $D=\begin{pmatrix}0 & qi\\ -qi & 0\end{pmatrix}$, *q real.*

$D^* = D$, so D is normal and self-adjoint. $DD^* = D^2 = \begin{pmatrix} q^2 & 0 \\ 0 & q^2 \end{pmatrix}$, so if $q = \pm 1$, D is unitary (but otherwise, it is not). D is not a projection, since $D^2 \neq D$.

E. $Tf(x) = \frac{1}{2}\left(f(x) + f(-x) \right)$.
T is self-adjoint (and therefore normal), since

$$\langle Tf, g \rangle = \frac{1}{2}\int_{-\infty}^{\infty} \left(f(x) + f(-x) \right) \overline{g(x)} dx = \frac{1}{2}\left(\int_{-\infty}^{\infty} f(x)\overline{g(x)} dx + \int_{-\infty}^{\infty} f(-x)\overline{g(x)} dx \right),$$

$$= \frac{1}{2}\left(\int_{-\infty}^{\infty} f(x)\overline{g(x)} dx + \int_{-\infty}^{\infty} f(y)\overline{g(-y)} dy \right) = \langle f, Tg \rangle$$

with the equality between lines justified by a change of variables $y = -x$ in the second integral

T replaces f by the even-symmetric component of f, so we expect that $T^2 = T$ (and it is):

$$T^2 f(x) = \frac{1}{2}\left(Tf(x) + Tf(-x) \right) = \frac{1}{2}\left(\frac{f(x) + f(-x)}{2} + \frac{f(-x) + f(x)}{2} \right) = \frac{f(x) + f(-x)}{2} = Tf(x).$$

So T is a projection (but not unitary).

F. $Wf(x) = xf(x)$.
W is self-adjoint (and therefore normal), since

$$\langle Wf, g \rangle = \int_{-\infty}^{\infty} xf(x)\overline{g(x)} dx = \int_{-\infty}^{\infty} f(x)\overline{xg(x)} dx = \langle f, Wg \rangle.$$

(x is real, so $\overline{x} = x$).

$W^2 f(x) = W(xf)(x) = x^2 f(x)$, so W is not unitary or a projection.

G. $Yf(x) = f'(x)$ ($f'(x) = \frac{df}{dx}$).
Y is not self-adjoint, since

$$\langle Yf, g \rangle = \int_{-\infty}^{\infty} f'(x)\overline{g(x)} dx = f(x)\overline{g(x)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\overline{g'(x)} dx = -\int_{-\infty}^{\infty} f(x)\overline{g'(x)} dx = -\langle f, Yg \rangle,$$ and

therefore, not unitary nor a projection. Here, we use integration by parts at the second step, and note that if f and g are square-integrable, then they must vanish at $\pm\infty$.
However, this does show that since $Y^* = -Y$, so $YY^* = Y^*Y = -Y^2$, and therefore Y is normal.

H. $Zf(x) = if'(x)$ ($f'(x) = \frac{df}{dx}$).
Z is self-adjoint, and therefore normal, since (as in G)

$$\langle Zf, g \rangle = \int_{-\infty}^{\infty} if'(x)\overline{g(x)}dx = -\int_{-\infty}^{\infty} if(x)\overline{g'(x)}dx = \int_{-\infty}^{\infty} f(x)\overline{ig'(x)}dx = \langle f, Zg \rangle .$$

($\bar{i} = -i$, absorbing the minus sign.)

$Z^2 f(x) = Z(if')(x) = i^2 f''(x) = -f''(x)$, so Z is not unitary or a projection.

Q2. Making self-adjoint operators and projections

Part A. For any operator A, show A^*A is self-adjoint.

Recall that the adjoint of a product is the product of the adjoints, in reverse order. Therefore, $\left(A^*A\right)^* = A^*\left(A^*\right)^*$.

Recall that the adjoint of the adjoint is the original transformation. So $\left(A^*\right)^* = A$, and,

$\left(A^*A\right)^* = A^*\left(A^*\right)^* = A^*A$.

Part B. Assuming that B^*B has an inverse, show
$P_B = B(B^*B)^{-1}B^*$ is a projection, by showing that it is idempotent and self-adjoint.

Idempotent:
$\left(P_B\right)^2 = \left(B(B^*B)^{-1}B^*\right)\left(B(B^*B)^{-1}B^*\right) = B(B^*B)^{-1}(B^*B)(B^*B)^{-1}B^* = B(B^*B)^{-1}B^* = P_B$.

Self-adjoint:
$\left(P_B\right)^* = \left(B(B^*B)^{-1}B^*\right)^* = B^{**}\left((B^*B)^{-1}\right)^* B^* = B\left((B^*B)^{-1}\right)^* B^*$ (second equality: adjoint of a product is the product of the adjoints in reverse order; third equality: $\left(B^*\right)^* = B$.)

Working on $\left((B^*B)^{-1}\right)^*$: $\left((B^*B)^{-1}\right)^* = \left((B^*B)^*\right)^{-1} = \left(B^*B\right)^{-1}$

(first equality: adjoint of an inverse is the inverse of the adjoint, second equality: part A.)