Linear Transformations and Group Representations

Homework #2 (2010-2011), Answers

Q1: Classifying some operators

In each case, determine whether the operators are normal, self-adjoint, unitary, or projections, using the standard inner product for a finite-dimensional space \((A, B, C, D)\), or for complex-valued functions on the line \((E, F, G, H)\). Note: \(G\) and \(H\) are a bit harder.

A. \(A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\).

\[ A^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \]

\[ AA^* = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

but \(A^*A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\). So \(A\) is not normal (or self-adjoint, or unitary, or a projection). But \(A^2 = A\), so \(A\) is idempotent.

B. \(B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & \sin \varphi \\ 0 & -\sin \varphi & \cos \varphi \end{pmatrix}\).

This is a rotation by angle \(\varphi\) in the \(e_2-e_3\)-plane. \(B^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix}\). \(BB^* = B^*B = I\), so \(B\) is normal and unitary (but not self-adjoint or a projection).

C. \(C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}\).

This is like image projection: the first two coordinates are averaged, and the last two coordinates are averaged. \(C^* = C\). So \(C\) is normal and self-adjoint. \(C^2 = C\), so \(C\) is a projection. \(CC^* = C^2 = C \neq I\), so \(C\) is not unitary.

D. \(D = \begin{pmatrix} 0 & qi \\ -qi & 0 \end{pmatrix}\), \(q\) real.
\( D^* = D \), so \( D \) is normal and self-adjoint. \( DD^* = D^2 = \begin{pmatrix} q^2 & 0 \\ 0 & q^2 \end{pmatrix} \), so if \( q = \pm 1 \), \( D \) is unitary (but otherwise, it is not). \( D \) is not a projection, since \( D^2 \neq D \).

**E.** \( Tf(x) = \frac{1}{2}(f(x) + f(-x)) \).

\( T \) is self-adjoint (and therefore normal), since

\[
\langle Tf, g \rangle = \frac{1}{2} \int_{-\infty}^{\infty} (f(x) + f(-x)) \overline{g(x)} \, dx = \frac{1}{2} \left( \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx + \int_{-\infty}^{\infty} f(-x) \overline{g(x)} \, dx \right)
\]

\[
= \frac{1}{2} \left( \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx + \int_{-\infty}^{\infty} f(y) \overline{g(-y)} \, dy \right) = \langle f, Tg \rangle
\]

with the equality between lines justified by a change of variables \( y = -x \) in the second integral.

\( T \) replaces \( f \) by the even-symmetric component of \( f \), so we expect that \( T^2 = T \) (and it is):

\[
T^2 f(x) = \frac{1}{2} \left( Tf(x) + Tf(-x) \right) = \frac{1}{2} \left( \frac{f(x) + f(-x)}{2} + \frac{f(-x) + f(x)}{2} \right) = \frac{f(x) + f(-x)}{2} = Tf(x).
\]

So \( T \) is a projection (but not unitary).

**F.** \( Wf(x) = xf(x) \).

\( W \) is self-adjoint (and therefore normal), since

\[
\langle Wf, g \rangle = \int_{-\infty}^{\infty} xf(x) \overline{g(x)} \, dx = \int_{-\infty}^{\infty} f(x) xg(x) \, dx = \langle f, Wg \rangle.
\]

(\( x \) is real, so \( \overline{x} = x \)).

\( W^2 f(x) = W(xf)(x) = x^2 f(x) \), so \( W \) is not unitary or a projection.

**G.** \( Yf(x) = f'(x) \) (\( f'(x) = \frac{df}{dx} \)).

\( Y \) is not self-adjoint, since

\[
\langle Yf, g \rangle = \int_{-\infty}^{\infty} f'(x) \overline{g(x)} \, dx = \int_{-\infty}^{\infty} f(x) \overline{g'(x)} \, dx = -\int_{-\infty}^{\infty} f(x) \overline{g'(x)} \, dx = -\langle f, Yg \rangle,
\]

and therefore, not unitary nor a projection. Here, we use integration by parts at the second step, and note that if \( f \) and \( g \) are square-integrable, then they must vanish at \( \pm \infty \).

However, this does show that since \( Y^* = -Y \), so \( YY^* = Y^*Y = -Y^2 \), and therefore \( Y \) is normal.

**H.** \( Zf(x) = if'(x) \) (\( f'(x) = \frac{df}{dx} \)).

\( Z \) is self-adjoint, and therefore normal, since (as in G)
\[ \langle Zf, g \rangle = \int_{-\infty}^{\infty} \overline{g(x)} f'(x) dx = -\int_{-\infty}^{\infty} f(x) \overline{g'(x)} dx = \int_{-\infty}^{\infty} f(x)ig'(x) dx = \langle f, Zg \rangle. \]

(\(i = -i\), absorbing the minus sign.)

\[ Z^2 f(x) = Z(if'(x)) = i^2 f''(x) = -f''(x), \] so \(Z\) is not unitary or a projection.

Q2. Making self-adjoint operators and projections

Part A. For any operator \(A\), show \(A^*A\) is self-adjoint.

Recall that the adjoint of a product is the product of the adjoints, in reverse order. Therefore, \((A^*A)^* = A^* (A^*)^*\).

Recall that the adjoint of the adjoint is the original transformation. So \((A^* )^* = A\), and, \((A^* A)^* = A^* (A^* )^* = A^* A\).

Part B. Assuming that \(B^* B\) has an inverse, show \(P_B = B(B^* B)^{-1} B^*\) is a projection, by showing that it is idempotent and self-adjoint.

Idempotent:

\[ (P_B)^2 = (B(B^* B)^{-1} B^*) (B(B^* B)^{-1} B^*) = B(B^* B)^{-1} (B^* B) (B^* B)^{-1} B^* = B(B^* B)^{-1} B^* = P_B. \]

Self-adjoint:

\[ (P_B)^* = (B(B^* B)^{-1} B^*)^* = B^* ((B^* B)^{-1})^* B^* = B((B^* B)^{-1})^* B^* \]

(second equality: adjoint of a product is the product of the adjoints in reverse order; third equality: \((B^*)^* = B\).)

Working on \((B^* B)^{-1})^* = ((B^* B)^{-1})^* = (B^* B)^{-1}\)

(first equality: adjoint of an inverse is the inverse of the adjoint, second equality: part A.)