Linear Transformations and Group Representations

Homework #3 (2010-2011), Answers

Q1: Representations of direct products of a group (or, two-dimensional Fourier transforms)

Setup: Given two groups, G_1 and G_2, we can define a group which is their "direct product" $G_1 \times G_2$, with elements (g_1, g_2) via the operation $(g_1, g_2) \circ (h_1, h_2) = (g_1 h_1, g_2 h_2)$. Moreover, given a representation L_1 of G_1 in vector space V_1 and a representation L_2 of G_2 in V_2, we can form a representation $L_1 \times L_2$ of $G_1 \times G_2$ in $V_1 \otimes V_2$, defined by $\left(L_1 \times L_2\right)_{(g_1, g_2)} = \left(L_1\right)_{g_1} \otimes \left(L_2\right)_{g_2}$. Given the above setup:

A. Find the character $\chi_{L_1 \times L_2}(g_1, g_2)$.

$$\chi_{L_1 \times L_2}(g_1, g_2) = \operatorname{tr}\left(\left(L_1 \times L_2\right)_{(g_1, g_2)}\right) = \operatorname{tr}\left(\left(L_1\right)_{g_1} \otimes \left(L_2\right)_{g_2}\right) = \operatorname{tr}\left(\left(L_1\right)_{g_1}\right) \operatorname{tr}\left(\left(L_2\right)_{g_2}\right) = \chi_{L_1}(g_1)\chi_{L_2}(g_2)$$

B. If G_1 and G_2 are finite, show that if L_1 is an irreducible representation of G_1 and L_2 is an irreducible representation of G_2, then $L_1 \times L_2$ is an irreducible representation of $G_1 \times G_2$.

According to the Group Representation Theorem, $L_1 \times L_2$ is irreducible if $d(L, L) = \frac{1}{|G|}\sum_g \left|\chi_L(g)\right|^2 = 1$ for $L = L_1 \times L_2$. Using part A to evaluate $\chi_{L_1 \times L_2}(g_1, g_2)$ and recognizing that a sum over all group elements in $G = G_1 \times G_2$ requires summing over all group elements g_1 in G_1 and g_2 in G_2 independently:

$$d(L_1 \times L_2, L_1 \times L_2) = \frac{1}{|G_1||G_2|}\sum_{g_1, g_2} \left|\chi_{L_1 \times L_2}(g_1, g_2)\right|^2 = \frac{1}{|G_1||G_2|}\sum_{g_1, g_2} \left|\chi_{L_1}(g_1)\chi_{L_2}(g_2)\right|^2 =$$
$$\frac{1}{|G_1||G_2|}\sum_{g_1, g_2} \left|\chi_{L_1}(g_1)\right|^2 \left|\chi_{L_2}(g_2)\right|^2 = \left(\frac{1}{|G_1|}\sum_{g_1} \left|\chi_{L_1}(g_1)\right|^2\right)\left(\frac{1}{|G_2|}\sum_{g_2} \left|\chi_{L_2}(g_2)\right|^2\right) = d(L_1, L_1)d(L_2, L_2).$$

Each of the factors in the final expression must be 1, since L_1 and L_2 are assumed to be irreducible.

Note that by a completely parallel computation, $d(L_1 \times L_2, M_1 \times M_2) = d(L_1, M_1)d(L_2, M_2)$. Tht is (and not surprisingly), each of the "factors" in a direct product can be considered completely independently.

Q2: Representations of the quaternion group

The "quaternion group" Q is defined as follows: It has 8 elements, ± 1, $\pm i$, $\pm j$, and $\pm k$. Group operations involving ± 1 are ordinary multiplication, for example, $(-1)j = -j$ and

$(-1)(-k)=k$. *Group operations not involving* ± 1 *are defined as follows:* $i^2=j^2=k^2=-1$, $ij=-ji=k$, $jk=-kj=i$, *and* $ki=-ik=j$, *with other products determined by the associative law, e.g.,* $(-i)j=-(ij)=-k$. *Note that* $x=\pm i$, *j, or k,* $x^4=1$ *and* $x^{-1}=x^3=x(x^2)=-x$, *so, for example,* $iji^{-1}=-iji=-ki=-j$. *If both x and* $y=\pm i$, *j, or k but* $x\neq \pm y$, *then x and y do not commute; in fact,* $xy=-yx$.

A. Find the conjugate classes of Q. (The conjugate class of an element g is the set of elements that are equivalent to g under inner automorphism, namely, the set of all elements $h^{-1}gh$.

The conjugate class of $+1$ only contains $+1$, since $+1$ commutes with everything. Similarly, the conjugate class of -1 only contains -1, since -1 commutes with everything.

Let $x=\pm i$, *j*, or *k* and similarly choose *h*. If $x=\pm h$, then *x* and *h* commute, and $h^{-1}xh=h^{-1}hx=x$. But if $x\neq \pm h$, then $xh=-hx$, so $h^{-1}xh=h^{-1}h(-1)x=-x$. So for $x=i$, *j*, or *k*, the conjugate class of *x* consists of *x* and $-x$.

B. Find the complete character table of Q.

Hint 1. We can construct a one-dimensional representation of Q as follows. Let $x=\pm i$ *,j, or k. For every element* y *of Q,* $x^{-1}yx$ *is either y or* $-y$. *So write* $s_x(y)=x^{-1}yxy^{-1}$, *where* $s_x(y)$ *is always* ± 1, *and hence commutes with everything. Show that* $s_x(yz)=s_x(y)s_x(z)$. *This means that for each* $x=\pm i$, *j, or k, the map from y to* $s_x(y)$ *is a one-dimensional representation.*

Hint 2. After constructing the above representations, determine which ones are different, and, by using the group representation theorem, determine that there is only one representation not yet accounted for – and find its character using the fact that the characters must be orthogonal.

We use a convention similar to that in the notes to write the character table and conjugate classes, and start with the identity representation:

	$+1$	-1	$i,-i$	$j,-j$	$k,-k$
E	1	1	1	1	1

.

Next use Hint 1. To show that the map from *g* to $s_x(g)$ is a representation, we need to verify that $s_x(yz)=s_x(y)s_x(z)$, which we can do as follows.

$$s_x(yz) = x^{-1}yzx(yz)^{-1} = \left(x^{-1}yx\right)\left(x^{-1}zx\right)z^{-1}y^{-1} = \left(s_x(y)y\right)\left(s_x(z)z\right)z^{-1}y^{-1}$$
$$= \left(s_x(y)y\right)\left(s_x(z)\right)y^{-1} = \left(s_x(y)yy^{-1}\right)\left(s_x(z)\right) = s_x(y)s_x(z).$$

Thus, for $x = \pm i$, $\pm j$, or $\pm k$, we have a one-dimensional (and hence irreducible) representation, in which g is mapped to $s_x(g)$. But $s_{-x}(g) = s_x(g)$, so we only get 3 such distinct representations, not 6. Since the "matrices" that form the representations are 1-dimensional (i.e., scalars), immediately one has $\chi_{s_x}(g) = s_x(g)$. This yields three lines in the character table:

	$+1$	-1	$i,-i$	$j,-j$	$k,-k$
E	1	1	1	1	1
s_i	1	1	1	-1	-1
s_j	1	1	-1	1	-1
s_k	1	1	-1	-1	1

.

Now use Hint 2. There can only be one more group representation, since its character must be orthogonal to the other 4 characters. It therefore must have dimension 2, since the sum of the squares of the dimensions of the representations must be equal to the size of the group. This is because the regular representation (which is of dimension 8, the size of the group) must contain each irreducible representation, and the number of occurrences of each is equal to its dimension; $8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$. (See comment concerning the cube example following the completed character table).

To find the character of this representation (say, *D*), note that it must be 0 for $x = \pm i$, $\pm j$, or $\pm k$, because the character of *D* must be orthogonal to all the other characters. For example, $\chi_E + \chi_{s_i} - \chi_{s_j} - \chi_{s_k}$ is equal to 0 except at $\pm i$ (where it is 4), so the unknown representation must have a character of 0 at $\pm i$. Since the *D* has a character of 2 at the identity, and a character of 0 at $x = \pm i$, $\pm j$, or $\pm k$, and its character must be orthogonal to that of the trivial representation, it follows that its character at -1 is -2:

	$+1$	-1	$i,-i$	$j,-j$	$k,-k$
E	1	1	1	1	1
s_i	1	1	1	-1	-1
s_j	1	1	-1	1	-1
s_k	1	1	-1	-1	1
D	2	-2	0	0	0

.

Note: the classical 2-d presentation of the quaternions corresponds to the representation *D*:
$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $k = ij = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.