Linear Transformations and Group Representations

Homework #3 (2010-2011)

Q1: Representations of direct products of a group (or, two-dimensional Fourier transforms)

Setup: Given two groups, G_1 and G_2, we can define a group which is their "direct product" $G_1 \times G_2$, with elements (g_1, g_2) via the operation $(g_1, g_2) \circ (h_1, h_2) = (g_1 h_1, g_2 h_2)$. Moreover, given a representation L_1 of G_1 in vector space V_1 and a representation L_2 of G_2 in V_2, we can form a representation $L_1 \times L_2$ of $G_1 \times G_2$ in $V_1 \otimes V_2$, defined by $\left(L_1 \times L_2\right)_{(g_1, g_2)} = \left(L_1\right)_{g_1} \otimes \left(L_2\right)_{g_2}$. Given the above setup:

A. Find the character $\chi_{L_1 \times L_2}(g_1, g_2)$.

B. If G_1 and G_2 are finite, show that if L_1 is an irreducible representation of G_1 and L_2 is an irreducible representation of G_2, then $L_1 \times L_2$ is an irreducible representation of $G_1 \times G_2$.

Q2: Representations of the quaternion group

The "quaternion group" Q is defined as follows: It has 8 elements, ± 1, $\pm i$, $\pm j$, and $\pm k$. Group operations involving ± 1 are ordinary multiplication, for example, $(-1)j = -j$ and $(-1)(-k) = k$. Group operations not involving ± 1 are defined as follows: $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$, with other products determined by the associative law, e.g., $(-i)j = -(ij) = -k$. Note that $x = \pm i$, j, or k, $x^4 = 1$ and $x^{-1} = x^3 = x(x^2) = -x$, so, for example, $iji^{-1} = -iji = -ki = -j$. If both x and $y = \pm i$, j, or k but $x \neq \pm y$, then x and y do not commute; in fact, $xy = -yx$.

A. Find the conjugate classes of Q. (The conjugate class of an element g is the set of elements that are equivalent to g under inner automorphism, namely, the set of all elements $h^{-1}gh$).

B. Find the complete character table of Q.

Hint 1. We can construct a one-dimensional representation of Q as follows. Let $x = \pm i$, j, or k. For every element y of Q, $x^{-1}yx$ is either y or $-y$. So write $s_x(y) = x^{-1}yxy^{-1}$, where $s_x(y)$ is always ± 1, and hence commutes with everything. Show that $s_x(yz) = s_x(y)s_x(z)$. This means that for each $x = \pm i$, j, or k, the map from y to $s_x(y)$ is a one-dimensional representation.

Hint 2. After constructing the above representations, determine which ones are different, and, by using the group representation theorem, determine that there is only one representation not yet accounted for – and find its character using the fact that the characters must be orthogonal.