Groups, Fields, and Vector Spaces

Homework #3 (2012-2013), Answers

(Q3 refers to first part of notes on "Linear Transformations and Group Representations")

Q1. Intrinsic relationships among dual spaces, etc.

A. Find an intrinsic relationship (a.k.a. "canonical homomorphism") between V and V^{**}. (V^{**} is the dual of V^*, i.e., the space of mappings from elements φ of V^* to the field.) That is, find a linear mapping Φ from elements v of V to elements $\Phi(v)$ of V^{**}.

To define $\Phi(v)$, we need to display $\Phi(v)$ as a linear map from elements φ of V^* to a field element. That is, we need to specify the field element $[\Phi(v)](\varphi)$ that φ is mapped to by $\Phi(v)$. (and ensure that everything is linear). Since φ is in V^*, $\varphi(v)$ is a linear map from *V* to the field. So we choose $[\Phi(v)](\varphi) = \varphi(v)$.

B. Find an intrinsic relationship (a.k.a. "canonical homomorphism") between $Hom(V,W)$ and $Hom(W^*,V^*)$. That is, find a linear mapping Z from elements φ of $Hom(V,W)$ to elements $Z(\varphi)$ of $Hom(W^*,V^*)$.

To define $Z(\varphi)$, we need to display $Z(\varphi)$ as a linear map from elements ζ of W^* to elements $[Z(\varphi)](\zeta)$ in V^*. That is, we need to define how $[Z(\varphi)](\zeta)$ acts on an element *v* of *V*. Since φ is in $Hom(V,W)$, $\varphi(v)$ is in *W*, and ζ acts linearly on it. So it is natural to define $\left([Z(\varphi)](\zeta)\right)(v) = \zeta(\varphi(v))$. Everything is linear.

C. Find an intrinsic relationship (a.k.a. "canonical isomorphism") between $(V \otimes W)^*$ and $Hom(V,W^*)$. That is, (a) given an element B of $(V \otimes W)^*$, find a linear mapping Φ that takes elements B of $(V \otimes W)^*$ to elements $\Phi(B)$ of $Hom(V,W^*)$. (b) Given an element ξ of $Hom(V,W^*)$, find a linear mapping Ψ that takes elements ξ of $Hom(V,W^*)$ to elements $\Psi(\xi)$ of $(V \otimes W)^*$. (c) Show that Φ and Ψ are inverses, i.e., $\Psi(\Phi(B)) = B$ and $\Phi(\Psi(\xi)) = \xi$.

(a) To define $\Phi(B)$, we have to show how it maps an element *v* of *V* into an element φ of W^*. That is, $[\Phi(B)](v)$ must be an element of W^*, i.e., a linear map from *W* to the scalars. So we have to show how $[\Phi(B)](v)$ acts on an arbitrary *w* in *W* (and everything has to be linear). Since *B* is given as an element of $(V \otimes W)^*$, it is a linear map from $v \otimes w$ to the field, just what we need. So $\Phi(B)$ is defined by $\left([\Phi(B)](v)\right)(w) = B(v \otimes w)$.

(b) To define $\Psi(\xi)$, we need to produce an element of $(V \otimes W)^*$, i.e., a linear map from tensors $v \otimes w$ to the field. Since ξ is in $Hom(V, W^*)$, $\xi(v)$ is in W^* and is therefore a map from *W* to the field, and $[\xi(v)](w)$ is linear in *v* and *w*. So we can take $[\Psi(\xi)](v \otimes w) = [\xi(v)](w)$.

(c) To show $\Psi(\Phi(B)) = B$: $[\Psi(\Phi(B))](v \otimes w) = \left([\Phi(B)](v)\right)(w) = B(v \otimes w)$. (First equality is from (b), second is from (a)). To show $\Phi(\Psi(\xi)) = \xi$:
$\left([\Phi(\Psi(\xi))](v)\right)(w) = (\Psi(\xi))(v \otimes w) = [\xi(v)](w)$. (First equality is from (a), second is from (b)).

Q2. Parity

A. What is the parity of a cyclic permutation of q elements, i.e., the permutation that puts 2 where 1 was, puts 3 where 2 was, puts 4 where 3 was, …, puts q where $q-1$ *was, and puts 1 where q was?*

This permutation can be generated by the following steps: swap 2 with 1, swap 3 with 1, swap 4 with 1, …, and swap *q* with 1. There are $q-1$ such steps, so the parity is $(-1)^{q-1}$.

In "permutation notation", this can be written $(123\ldots q) = (12)(13)(14)\ldots(1q)$, where $(abc\cdots ef)$ means "put *b* where *a* was, put *c* where *b* was, put *f* where *e* was, and put *a* where *f* was"

B. Recall the dihedral group: the symmetry group of a regular n-gon, containing rotations by $2\pi k/n$ *radians, and reflections. (a) It can also be considered a permutation group, because it permutes the vertices of the n-gon. Which group elements correspond to a permutation with an even parity, and which to an odd parity? (b) The dihedral group can also be considered a permutation group in another way, because it acts on the edges of the n-gon. – i.e., a rotation or a reflection of the dihedral group is a permutation on the edges. In this representation, which group elements correspond to permutations with even parity, and which ones to an odd parity?*

(a) A rotation by $2\pi/n$ radians (one "step") is a cyclic permutation of the *n* vertices, and its parity (as in part A) is therefore $(-1)^{n-1}$. So a rotation by $2\pi k/n$ radians has parity $\left((-1)^{(n-1)}\right)^k = (-1)^{k(n-1)}$.

The behavior of reflections depends on whether *n* is even or *n* is odd. If *n* is odd, then every reflection goes through one vertex, and the other $n-1$ vertices are swapped in pairs. So every reflection has a parity of $(-1)^{(n-1)/2}$. If *n* is even, then there are two kinds of reflections: those whose mirror planes go through the midpoints of opposite edges, and those whose mirror planes go through opposite vertices. The ones whose mirror planes go through the midpoints of opposite edges result in swapping the *n* vertices in pairs, and thus, have a parity of $(-1)^{n/2}$. The ones whose mirror planes go through opposite vertices leave those two vertices unchanged, and swap the remaining *n*-2 vertices in pairs. They therefore have a parity of $(-1)^{(n-2)/2}$.

(b) Rotations can be analyzed just as above; the one-step rotation is a cyclic permutation on the *n* edges.

For reflections: When *n* is odd, the analysis of part (a) holds, since one can track each edge by what happens to the opposite vertex. When *n* is even, the mirror planes that go through the midpoints of opposite edges have parity $(-1)^{(n-2)/2}$ (since they leave two edges unchanged), and the ones that go through opposite vertices have parity $(-1)^{n/2}$, since they swap all edges.

Comment:

This shows that parity is not an intrinsic property of the group element, only of how it is represented as a permutation group. (For *n* even, reflections have opposite parities, depending on whether they are considered to act on vertices or on edges.)

Also, it allows us to find some subgroups of the dihedral group: group elements that have even parity are the kernel of the mapping from the group to $\{+1,-1\}$ via the *parity* homomorphism.

Q3. Eigenvectors and Eigenvalues

Setup common to all parts: A is in $Hom(V,V)$*, where V is of dimension m, and it has m eigenvectors* v_j *and m corresponding eigenvalues* λ_j*;* $Av_j = \lambda_j v_j$*. B is in* $Hom(W,W)$*, where W is of dimension n, and it has n eigenvectors* w_j *and n corresponding eigenvalues* μ_k*;* $Bw_k = \mu_k w_k$*. We assume that all eigenvalues are distinct.*

A. We can define a linear transformation $A\oplus B$ *in* $Hom(V\oplus W, V\oplus W)$ *by its action on elements of* $V\oplus W$*, namely,* $\left(A\oplus B\right)(v,w)=(Av,Bw)$*. What are the eigenvectors and eigenvalues of* $A\oplus B$*? What is its trace? What is its determinant?*

Here and in the other parts, we guess the eigenvectors (and verify the guess from the setup), and show that we have a complete set by counting them. Then we compute the eigenvalues from the setup; the trace is the sum of the eigenvalues and the determinant is their product.

Eigenvectors of $A\oplus B$: The *m* vectors $(v_j,0)$ and the *n* vectors $(0,w_k)$.
Verification:
$\left(A\oplus B\right)(v_j,0)=(Av_j,B0)=(\lambda_j v_j,0)=\lambda_j(v_j,0)$.
$\left(A\oplus B\right)(0,w_k)=(0,Bw_k)=(0,\mu_k w_k)=\mu_k(0,w_k)$.
So the eigenvalues are $\left\{\lambda_1,\ldots,\lambda_m,\mu_1,\ldots,\mu_n\right\}$.
The trace is their sum, $\lambda_1+\ldots+\lambda_m+\mu_1+\ldots+\mu_n=\mathrm{tr}(A)+\mathrm{tr}(B)$.
The determinant is their product, $\lambda_1\ldots\lambda_m\mu_1\ldots\mu_n=\det(A)\det(B)$.

B. We can define a linear transformation $A\otimes B$ in $Hom(V\otimes W, V\otimes W)$ by its action on the elementary tensor products $v\otimes w$ in $V\otimes W$, namely, $(A\otimes B)(v\otimes w)=Av\otimes Bw$ (as in notes). What are the eigenvectors and eigenvalues of $A\otimes B$? What is its trace? What is its determinant?

Eigenvectors of $A\otimes B$: The *mn* vectors $v_j\otimes w_k$.

Verification:
$(A\otimes B)(v_j\otimes w_k)=Av_j\otimes Bw_k=(\lambda_j v_j)\otimes(\mu_k w_k)=(\lambda_j\mu_k)(v_j\otimes w_k)$.
So the eigenvalues are all of the products $\{\lambda_1\mu_1,\ldots,\lambda_m\mu_1,\ldots,\lambda_1\mu_n,\ldots,\lambda_m\mu_n\}$.
The trace is their sum,
$\lambda_1\mu_1+\ldots+\lambda_m\mu_1+\ldots+\lambda_1\mu_n+\ldots+\lambda_m\mu_n=(\lambda_1+\ldots+\lambda_m)(\mu_1+\ldots+\mu_n)=(\mathrm{tr}\,A)(\mathrm{tr}\,B)$.
The determinant is their product,
$(\lambda_1\mu_1)\ldots(\lambda_m\mu_1)\ldots(\lambda_1\mu_n)\ldots(\lambda_m\mu_n)=(\lambda_1\ldots\lambda_m)^n(\mu_1\ldots\mu_n)^m=(\det A)^n(\det B)^m$ (each λ occurs *n* times, since it has to be paired with each μ; similarly, each μ occurs *m* times, since it has to be paired with each λ.

C. As a special case of part B, we can take $W=V$ and $B=A$. So now $A\otimes A$ has an action in $V\otimes V$, and also in its subspaces $sym(V^{\otimes 2})$ and $anti(V^{\otimes 2})$. Let $A\otimes_{sym}A$ denote the action of $A\otimes A$ in $sym(V^{\otimes 2})$, and, similarly, for $A\otimes_{anti}A$. What are the eigenvectors and eigenvalues of $A\otimes_{sym}A$ and $A\otimes_{anti}A$?

In $sym(V^{\otimes 2})$: The eigenvectors of $A\otimes_{sym}A$ are the symmetrized versions of the vectors $v_j\otimes v_k$, namely, $sym(v_j\otimes v_k)=\frac{1}{2}(v_j\otimes v_k+v_k\otimes v_j)$. This holds because $v_j\otimes v_k$ and $v_k\otimes v_j$ have the same eigenvalue for $A\otimes A$, namely, $\lambda_j\lambda_k$. Note that $sym(v_j\otimes v_k)=sym(v_k\otimes v_j)$, and we don't want to count any eigenvector twice. So there are $\frac{1}{2}m(m-1)$ eigenvectors $sym(v_j\otimes v_k)$ (for $j<k$), and *m* eigenvectors $sym(v_j\otimes v_j)$. This is a total of $\frac{1}{2}m(m+1)$ eigenvectors, which is the dimension of $sym(V^{\otimes 2})$.

In $anti(V^{\otimes 2})$: As above, the eigenvectors of $A\otimes_{anti}A$ are the antisymmetrized versions of the vectors $v_j\otimes v_k$, namely, $anti(v_j\otimes v_k)=\frac{1}{2}(v_j\otimes v_k-v_k\otimes v_j)$. Note that $anti(v_j\otimes v_k)=-anti(v_k\otimes v_j)$, and we don't want to count any eigenvector twice. So there are

$\frac{1}{2}m(m-1)$ eigenvectors $anti(v_j \otimes v_k)$ (for $j<k$). The quantities $anti(v_j \otimes v_j)$ are all zero, so they don't yield any new eigenvectors. This is a total of $\frac{1}{2}m(m-1)$ eigenvectors, which is the dimension of $anti(V^{\otimes 2})$.

D. What are the eigenvectors and eigenvalues of A^2 ?

Eigenvectors of A^2: The *m* vectors v_j.
Verification:
$A^2 v_j = A(Av_j) = A(\lambda_j v_j) = \lambda_j A v_j = \lambda_j^2 v_j$.
So the eigenvalues are $\{\lambda_1^2, \ldots, \lambda_m^2\}$.

Comment 1:

Part D allows us to write formulas for the traces of the operators in Part C:

$$\mathrm{tr}(A \otimes_{sym} A) = \sum_{j=1}^{m} \lambda_j^2 + \sum_{j<k} \lambda_j \lambda_k = \frac{1}{2}\left(\sum_{j=1}^{m} \lambda_j^2 + 2\sum_{j<k} \lambda_j \lambda_k + \sum_{j=1}^{m} \lambda_j^2\right) = \frac{1}{2}\left(\left(\sum_{j=1}^{m} \lambda_j\right)^2 + \sum_{j=1}^{m} \lambda_j^2\right) = \frac{\left(\mathrm{tr}(A)\right)^2 + \mathrm{tr}(A^2)}{2}$$

$$\mathrm{tr}(A \otimes_{anti} A) = \sum_{j<k} \lambda_j \lambda_k = \frac{1}{2}\left(\sum_{j=1}^{m} \lambda_j^2 + 2\sum_{j<k} \lambda_j \lambda_k - \sum_{j=1}^{m} \lambda_j^2\right) = \frac{1}{2}\left(\left(\sum_{j=1}^{m} \lambda_j\right)^2 - \sum_{j=1}^{m} \lambda_j^2\right) = \frac{\left(\mathrm{tr}(A)\right)^2 - \mathrm{tr}(A^2)}{2}$$

Comment 2:

It is evident that for A^p, the eigenvectors are the *m* vectors v_j, and the eigenvalues are $\{\lambda_1^p, \ldots, \lambda_m^p\}$. Working formally, this motivates a definition of $f(A)$ for any analytic function *f*, whenever the eigenvectors of *A* span *V*: $f(A)$ is defined as the linear transformation whose eigenvectors are the v_j, and whose eigenvalues are $\{f(\lambda_1), \ldots, f(\lambda_m)\}$. We then extend $f(A)$ by linearity from the set of eigenvectors to the full space *V*: if $v = \sum_j a_j v_j$, then

$\left(f(A)\right)v = \sum_j a_j f(\lambda_j) v_j$.