Graph-Theoretic Methods

Homework #2 (2012-2013), Answers

Q1. Consider the star graph with $N+1$ *vertices (one central vertex, each connected to the N peripheral vertices). Use symmetry considerations to find the eigenvalues and eigenvectors of its graph Laplacian.*

The full permutation group on *N* elements has a representation in the vector space of functions on the graph, via relabeling of the *N* peripheral vertices. So every eigenvector of the Laplacian must lie entirely within one of its irreducible subspaces. The functions on the graph are a space *V* of dimension $N+1$. There is an "obvious" two-dimensional subspace that is left invariant by these mappings – the subspace of functions that have a value x_c on the central vertex, and another value x_p on all peripheral nodes. This contains the eigenvector with an eigenvalue of zero (the vector $\vec{1}$, corresponding to $x_c = x_p = 1$).

This invariant subspace must contain another eigenvector, orthogonal to $\vec{1}$. Orthogonality requires that this eigenvector $\vec{w}$ must have coordinates proportional to $x_c = N$ and $x_p = -1$. The Laplacian maps the value at the central node to $x_c{}' = Nx_c - Nx_p$, and the value at a peripheral node to $x_p{}' = x_p - x_c$. With $x_c = N$ and $x_p = -1$, $x_c{}' = N^2 + N = (N+1)x_c$, and $x_p{}' = -1 - N = (1+N)x_p$, so $L\vec{w} = (N+1)\vec{w}$, and the eigenvalue is $N+1$.

Since *V* is of dimension $N+1$, there are $N-1$ dimensions unaccounted for. This corresponds to the $(N-1)$-dimensional irreducible representation of the symmetric group. We can see this by considering how the permutation group acts on the *N* peripheral vertices – it is just like the way it acts on the complete graph, an *N*-dimensional representation that contains one copy of the identity representation(corresponding to $\vec{w}$), and an irreducible $N-1$ -dimensional representation. The $N-1$-dimensional representation lies in a subspace in which the values at the peripheral vertices sum to zero: for example, a vector $\vec{y}$ with $x_1 = 1$, $x_2 = -1$, and the remainder zero. One can readily calculate that $L\vec{y} = \vec{y}$.

As a check, (see Q1 of GTM Homework #1), we have one eigenvalue equal to $N+1$, and $N-1$ eigenvalues equal to 1, for a total of $2N$, which is twice the number of edges in the star graph.