Linear Systems, Black Boxes, and Beyond

Homework #1 (2012-2013), Answers

Q1: Fourier transforms, derivatives, and integrals

Setup is $\hat{s}(\omega) = \int_{-\infty}^{\infty} s(t)e^{-i\omega t} dt$, with $s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{s}(\omega)e^{+i\omega t} d\omega$.

A. For $q(t) = \frac{d}{dt} s(t)$, find $\hat{q}(\omega)$.

Using the “synthesis” integral,

\begin{align*}
q(t) &= \frac{d}{dt} s(t) = \frac{d}{dt} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{s}(\omega)e^{+i\omega t} d\omega \right) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{s}(\omega) \frac{d}{dt} \left( e^{+i\omega t} \right) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{s}(\omega) (i\omega e^{+i\omega t}) d\omega.
\end{align*}

So the coefficient of $e^{i\omega t}$ in $q(t) = \frac{d}{dt} s(t)$ is $\hat{q}(\omega) = i\omega \hat{s}(\omega)$.

B. For $q_n(t) = \frac{d^n}{dt^n} s(t)$, find $\hat{q}_n(\omega)$.

Iterating part A: $\hat{q}_n(\omega) = i\omega \hat{q}_{n-1}(\omega)$, so $\hat{q}_n(\omega) = (i\omega)^n \hat{s}(\omega)$.

C. For $z(t) = \int_{-\infty}^{t} s(\tau)d\tau$, find $\hat{z}(\omega)$.

Since $s(t) = \frac{dz}{dt}$, we can use part A: $\hat{z}(\omega) = i\omega \hat{s}(\omega)$, so, except possibly at $\omega = 0$, $\hat{z}(\omega) = \frac{\hat{s}(\omega)}{i\omega}$.

D. Apply C to $s(t) = \delta(t)$ to find a function whose Fourier transform, except possibly at 0, is $\frac{1}{i\omega}$.

Since the Fourier transform of the delta-function is 1 everywhere, the integral of the delta-function, $h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$ has the required Fourier transform $\frac{1}{i\omega}$. Since the delta-function is an infinitesimally narrow peak with a unit area, the integral evaluates as $h(t) = \begin{cases} 1, t > 0 \\ 0, t < 0 \end{cases}$. This is the “Heaviside step function.” Its value at zero, which is formally undefined, is irrelevant for most purposes.
Q2: Fourier transforms and moments

Setup is \( \hat{s}(\omega) = \int_{-\infty}^{\infty} s(t)e^{-i\omega t} \, dt \), with \( s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{s}(\omega)e^{+i\omega t} \, d\omega \), but now we are thinking of \( s \) as a probability distribution.

A. Write the normalization condition \( \int_{-\infty}^{\infty} s(t)dt = 1 \) in terms of \( \hat{s}(\omega) \).

Since \( e^{i\omega t} = 1 \) for \( \omega = 0 \), \( \hat{s}(0) = \int_{-\infty}^{\infty} s(t)dt \), so the normalization condition is \( \hat{s}(0) = 1 \).

B. Write the mean (first moment) \( \langle t \rangle = \int_{-\infty}^{\infty} ts(t)dt \) in terms of \( s'(\omega) = \frac{d}{d\omega} \hat{s}(\omega) \).

Since \( \hat{s}'(\omega) = \int_{-\infty}^{\infty} s(t)\frac{d}{d\omega}e^{-i\omega t} \, dt = \int_{-\infty}^{\infty} s(t)(-it)e^{-i\omega t} \, dt \), it follows that \( \hat{s}'(0) = \int_{-\infty}^{\infty} s(t)(-it)dt \) and that \( \int_{-\infty}^{\infty} ts(t)dt = is'(0) \).

C. Write the variance (second moment) \( \left( (t - \langle t \rangle)^2 \right) = \langle t^2 \rangle - \langle t \rangle^2 = \int_{-\infty}^{\infty} t^2s(t)dt - \left( \int_{-\infty}^{\infty} ts(t)dt \right)^2 \) in terms of \( s'(\omega) = \frac{d}{d\omega} \hat{s}(\omega) \) and \( s''(\omega) = \frac{d^2}{d\omega^2} \hat{s}(\omega) \).

As in part B, \( \hat{s}''(\omega) = \int_{-\infty}^{\infty} s(t)\frac{d^2}{d\omega^2}e^{-i\omega t} \, dt = \int_{-\infty}^{\infty} s(t)(-t^2)e^{-i\omega t} \, dt \), so \( \int_{-\infty}^{\infty} t^2s(t)dt = -\hat{s}''(0) \).

So \( \int_{-\infty}^{\infty} t^2s(t)dt - \left( \int_{-\infty}^{\infty} ts(t)dt \right)^2 = -\hat{s}''(0) - (is'(0))^2 = -\hat{s}''(0) + (s'(0))^2 \).

Q3: The half-infinite cable (repeating indefinitely to the right)
This is to be viewed as a network of resistors and capacitors. Calculate the impedance of the system (input applied across terminals at left) in terms of the impedances \( F(\omega) \), \( G_1(\omega) \), and \( G_2(\omega) \) for \( F \), \( G_1 \), and \( G_2 \).

**Hint:** Let the composite system be \( H \). Note the following, and then write an equation for \( H(\omega) \).

\[
H = \frac{G_1}{F + H} + G_2
\]

The impedance of the composite system on the left is a series combination of three components: \( G_1 \), the parallel combination of \( F \) and \( H \), and \( G_2 \). Therefore its impedance is

\[
G_1(\omega) + \frac{F(\omega)H(\omega)}{F(\omega) + H(\omega)} + G_2(\omega).
\]

Since (as the hint indicates) this is equivalent to the entire half-infinite cable, \( H(\omega) = G_1(\omega) + \frac{F(\omega)H(\omega)}{F(\omega) + H(\omega)} + G_2(\omega) \). Solving for \( H(\omega) \) yields

\[
H(\omega)^2 - G(\omega)H(\omega) - G(\omega)F(\omega) = 0,
\]

where \( G(\omega) = G_1(\omega) + G_2(\omega) \), or,

\[
H(\omega) = \frac{G(\omega) + \sqrt{G(\omega)^2 + 4F(\omega)G(\omega)}}{2}.
\]

Note concerning the continuum limit: This corresponds to allowing each subunit to represent progressively less and less length. Then \( F \) has units of impedance/cm (and increases as the segment shortens), and \( G \) has units of impedance-cm (and decreases as the segment shortens). In this limit, \( H(\omega) \approx \sqrt{F(\omega)G(\omega)} \). This enables one to calculate the “cable length” \( \lambda \), which is the distance required for the transmembrane current to fall by a factor of \( e \). To do this, note that total transmembrane current \( I_{\text{total}} \) is \( \int_0^\infty e^{-x/\lambda} dx = \lambda \) times the current per unit length \( I_{\text{peak}} \) at the injection site, but also, \( I_{\text{total}} / I_{\text{peak}} \) is inversely proportional to the total cable impedance \( H(\omega) \), divided by the impedance per unit length, \( F(\omega) \). So \( \lambda = \frac{H(\omega)}{F(\omega)} = \frac{G(\omega)}{F(\omega)} \).

**Q4. Boxcar smoothing**

Boxcar smoothing refers to convolution with the function \( s(t) \), where \( s(t) = \begin{cases} \frac{1}{L}, & |t| \leq L/2 \\ 0, & |t| > L/2 \end{cases} \). Find its Fourier transform. What does it look like? Is this a good way to smoothe?
\[
\hat{s}(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt = \frac{1}{L} \int_{-L/2}^{L/2} e^{-i\omega t} dt = \frac{1}{-i\omega L} e^{-i\omega t/2} \bigg|_{-L/2}^{L/2} = \frac{e^{i\omega L/2} - e^{-i\omega L/2}}{i\omega L} = \frac{\sin(\omega L/2)}{\omega L/2}.
\]

This (the “sinc” function) has a peak of 1 at \( \omega = 0 \), and descends in an envelope proportional to \( 1/|\omega| \) away from zero. There are zeros at \( \omega = 2\pi k / L \), for \( k \neq 0 \). The center lobe (at \( \omega = 0 \)) is positive, but the adjacent lobes (\( \frac{2\pi}{L} < |\omega| < \frac{4\pi}{L} \)) are negative. So one problem with using this as a smoothing function is that it inverts the phase of non-negligible frequency components.

```matlab
>> x=[-8:0.01:8];
>> y=sinc(pi*x);
>> plot(x,y)
>> hold on;
>> plot([-8 8],[0 0],'k')
>> plot([0 0],[-0.5 1],'k')
>> set(gca,'YLim',[-0.2 1])
>> set(gca,'YLim',[-0.25 1])
>> xlabel('omega, as a multiple of 2pi/L')
>> set(gca,'XTick',[-8:8])
```