Linear Systems, Black Boxes, and Beyond

Homework #2 (2012-2013), Answers

Q1: Some standard windowing functions. For each of the windowing functions $W(t)$ below (each nonzero only on $[-L/2, L/2]$), calculate their Fourier transforms $\hat{W}(\omega) = \int\limits_{-\infty}^{\infty} W(t) e^{-i\omega t} dt$ *and characterize the asymptotic behavior of* $\left|\hat{W}(\omega)\right|^2$ *for large* $|\omega|$*. As mentioned in class, each windowing function represents a tradeoff between the sharpness of the peak of* $\left|\hat{W}(\omega)\right|^2$ *and the heaviness of its tails.*

A. $W_{boxcar}(t) = \frac{1}{L}$ *for* $|t| \le L/2$*, 0 otherwise*

B. $W_{tent}(t) = \frac{2}{L}(1 - \frac{2|t|}{L})$ *for* $|t| \le L/2$*, 0 otherwise*

C. $W_{cosinebell}(t) = \frac{1}{L}\left(1 + \cos\left(\frac{2\pi t}{L}\right)\right)$ *for* $|t| \le L/2$*, 0 otherwise*

In each case, W is an even-symmetric function, so we can write

$$\hat{W}(\omega) = \int\limits_{-\infty}^{\infty} W(t) e^{-i\omega t} dt = 2\int\limits_{0}^{\infty} W(t)\cos(\omega t) dt = 2\int\limits_{0}^{L/2} W(t)\cos(\omega t) dt .$$

For A,

$$\hat{W}_{boxcar}(\omega) = \frac{2}{L}\int\limits_{0}^{L/2} \cos(\omega t) dt = \frac{2}{\omega L}\sin(\omega t)\Big|_{0}^{L/2} = \frac{2}{\omega L}\sin(\frac{\omega L}{2}),$$ so $\left|\hat{W}_{boxcar}(\omega)\right|^2 \propto |\omega|^{-2}$ for large $|\omega|$.

For B,

$$\hat{W}_{tent}(\omega) = \frac{4}{L}\int\limits_{0}^{L/2}\left(1 - \frac{2t}{L}\right)\cos(\omega t) dt = \frac{4}{L}\left(\frac{1}{\omega}\sin(\omega t) - \frac{2t}{\omega L}\sin(\omega t) - \frac{2}{\omega^2 L}\cos(\omega t)\right)\Bigg|_{0}^{L/2}$$

$$= \frac{8}{\omega^2 L^2}\left(1 - \cos(\frac{\omega L}{2})\right) = \frac{16}{\omega^2 L^2}\sin^2(\frac{\omega L}{4})$$

, using $1 - \cos\phi = 2\sin^2\frac{\phi}{2}$ at the last step, so $\left|\hat{W}_{tent}(\omega)\right|^2 \propto |\omega|^{-4}$ for large $|\omega|$.

Note that $W_{tent}(t) = \left(2W_{boxcar} * 2W_{boxcar}\right)(2t)$, i.e., the tent a rescaled convolution of the boxcar with itself. So we can check: $\hat{W}_{tent}(\omega) = \left(\hat{W}_{tent}(\frac{\omega}{2})\right)^2$.

For C,

$$\hat{W}_{cosinebell}(\omega) = \frac{2}{L}\int_0^{L/2}\left(1+\cos\left(\frac{2\pi t}{L}\right)\right)\cos(\omega t)dt = \frac{2}{L}\int_0^{L/2}\left(\cos(\omega t)+\frac{1}{2}\cos\left(\left(\frac{2\pi}{L}+\omega\right)t\right)+\frac{1}{2}\cos\left(\left(\frac{2\pi}{L}-\omega\right)t\right)\right)dt$$

$$= \frac{2}{L}\left(\frac{\sin(\omega t)}{\omega}+\frac{1}{2\left(\frac{2\pi}{L}+\omega\right)}\sin\left(\left(\frac{2\pi}{L}+\omega\right)t\right)+\frac{1}{2\left(\frac{2\pi}{L}-\omega\right)}\sin\left(\left(\frac{2\pi}{L}-\omega\right)t\right)\right)\Bigg|_0^{L/2}$$

$$= \frac{2}{L}\left(\frac{\sin(\omega L/2)}{\omega}+\frac{1}{2\left(\frac{2\pi}{L}+\omega\right)}\sin\left(\pi+\omega L/2\right)+\frac{1}{2\left(\frac{2\pi}{L}-\omega\right)}\sin\left(\pi-\omega L/2\right)\right)$$

$$= \frac{2}{L}\left(\frac{\sin(\omega L/2)}{\omega}+\frac{1}{2}\sin\left(\omega L/2\right)\left(\frac{1}{\left(\frac{2\pi}{L}-\omega\right)}-\frac{1}{\left(\frac{2\pi}{L}+\omega\right)}\right)\right)$$

$$= \frac{\sin(\omega L/2)}{\omega L/2}\left(1+\frac{\omega}{2}\left(\frac{1}{\left(\frac{2\pi}{L}-\omega\right)}-\frac{1}{\left(\frac{2\pi}{L}+\omega\right)}\right)\right) = \frac{\sin(\omega L/2)}{\omega L/2}\left(1+\left(\frac{\omega^2}{\left(\frac{2\pi}{L}\right)^2-\omega^2}\right)\right)$$

$$= \frac{\sin(\omega L/2)}{\omega L/2}\left(\frac{\left(\frac{2\pi}{L}\right)^2}{\left(\frac{2\pi}{L}\right)^2-\omega^2}\right) = \frac{\sin(\omega L/2)}{\omega L/2}\left(\frac{1}{1-\left(\omega L/2\pi\right)^2}\right)$$

so $\left|\hat{W}_{cosinebell}(\omega)\right|^2 \propto |\omega|^{-6}$ for large $|\omega|$.