Linear Transformations and Group Representations

Homework #1 (2012-2013)

Q1: Eigenvectors of some linear operators in matrix form (also see Homework from “Algebraic Overview” (2008-2009))

In each case, find the eigenvalues, the eigenvectors, the dimensions of the eigenspaces, and whether a basis can be chosen from the eigenvectors.

A. \[ A = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}. \]

B. \[ B = \begin{pmatrix} q & r \\ r & q \end{pmatrix} \text{ (assume } q > r > 0). \]

C. \[ C = \begin{pmatrix} q & r \\ -r & q \end{pmatrix} \]

Q2: Eigenvectors of some linear operators in a continuous space

\( V \) is a vector space of functions of time. In each case, find the eigenvalues and eigenvectors of the indicated operator, and determine whether the operator is time-translation invariant.

A. \( L v(t) = tv(t) \).

B. \( R v(t) = v(-t) \).

C. \( M v(t) = \frac{d}{dt} v(t) \).

Q3: Knowing vector lengths determines the inner product.

\( V \) is a Hilbert space, and \( \langle v, w \rangle \) is its inner product. Write \( \langle v, w \rangle \) in terms of the squared vector lengths \( \| av + bw \|^2 = \langle av + bw, av + bw \rangle \) for selected values of \( a \) and \( b \). Hint: consider especially \( (a,b) = \{(1,1), (1,-1), (1,i), (1,-i)\} \).