Linear Transformations and Group Representations

Homework #2 (2012-2013)

Q1: Demonstrating that the pseudoinverse construction yields a projection.

The notes stated that one could construct a projection onto the range of an operator B as $P_B = B(B^*B)^{-1}B^*$, with the fine print that the inverse of B^*B is only computed within the range of B. Show that P_B is a projection, i.e., show that it is self-adjoint and that $P_BP_B = P_B$.

Q2: Another example of a group representation.

Consider the permutations of a set of $n=3$ abstract elements, $S=\{a,b,c\}$. There are $n!=6$ permutations of these 3 elementss, and they form a group, *G*, under composition. Let *V* be the 3-dimensional vector space *V* of functions on these elements. We can define a unitary representation of *G* in $Hom(V,V)$ as follows: The unitary transformation U_σ corresponding to the permutation σ is the transformation that takes the function f to $U_\sigma(f)$, where $\left(U_\sigma f\right)(x) = f(\sigma^{-1}(x))$, where $\sigma^{-1}(x)$ denotes the element of *S* that is moved to *x* by σ.

A. Verify that this is a representation. That is, show that composition of permutations σ and τ corresponds to composition of the corresponding transformations U_σ and U_τ, $U_\sigma U_\tau = U_{\sigma\tau}$.

B. Choose a basis set for *V*, as follows: $f_a(x) = \begin{cases} 1, & x=a \\ 0, & x \neq a \end{cases}$, and similarly for f_b and f_c. So for any *f*, $f = f(a)f_a + f(b)f_b + f(c)f_c$, i.e. $f = \begin{pmatrix} f(a) \\ f(b) \\ f(c) \end{pmatrix}$. In this basis, write the matrix form of U_σ for $\sigma=(ab)$ (σ is the permutation that takes *a* to *b* and *b* to *a*) and U_τ for $\tau=(abc)$ (τ is the permutation that takes *a* to *b, b* to *c*, and *c* to *a*).