Exam, 2012-2013

Do a total of 12 points or any four complete questions. Show your work!

Q1: 5 parts/5 points, no dependencies
Q2: 3 parts/3 points, no dependencies
Q3: 3 parts/3 points, parts B and C depend on A
Q4: 2 parts/2 points, no dependencies
Q5: 2 parts/2 points, part B depends on A
Q6: 2 parts/2 points, part B depends on A

Q1. Semidirect products (5 parts, 1 point for each part)

Here we define the “semi-direct product,” a standard way of building larger groups from smaller ones. Let \( H \) and \( K \) be groups, with a homomorphism from \( K \) into the automorphism group of \( H \). That is, for each element \( k \in K \), there is an automorphism \( \alpha_k \) of \( H \), and \( \alpha_k \alpha_k' = \alpha_{kk'} \).

With this setup, we can define an operation on \((h,k)\) pairs: \((h,k) \circ (h',k') = (h\alpha_k(h'), kk')\), where the composition \( h\alpha_k(h') \) takes place in \( H \), and the composition \( kk' \) takes place in \( K \).

A. Show that this operation forms a group, known as the “semidirect product of \( H \) and \( K \)”.

B. Recall the definition of a normal subgroup: A subgroup \( N \) of \( G \) is said to be a “normal” subgroup if, for any element \( g \) of \( G \) and any element \( n \) of \( N \), the combination \( gng^{-1} \) is also a member of \( N \). Determine whether the set of elements \( S_K = \{(e_H,k)\} \) is a normal subgroup.

C. Determine whether the set of elements \( S_H = \{(h,e_k)\} \) is a normal subgroup.

D. Use the above construction to create a continuous non-commutative group, where \( H \) and \( K \) are both commutative.

E. Use the above construction to create a discrete non-commutative group that is of size 21.
Q2. Operator exponentials (3 parts, 1 point for each part)

Setup: $A$ is a Hermitian operator; $s$ and $t$ are complex numbers. Define $e^{As} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$, where $A^k$ indicates (ordinary) repetitive application of the operator $A$ (and $A^0 = I$).

A. Is $e^{A(s+t)} = e^{As} e^{At}$? Why or why not?
B. Is $e^{(A+B)s} = e^{As} e^{Bs}$? Why or why not?
C. What is $\det(e^A)$ in terms of the coefficients of the characteristic equation for $A$?
Q3. Transfer functions and power spectra (3 parts, 1 point for each part)

The boxes are linear filters, with transfer functions $\tilde{F}_i(\omega)$ and $\tilde{H}_i(\omega)$.

A. Determine the relationships between the Fourier transforms of the inputs $\tilde{s}_i(\omega)$ and the outputs $\tilde{r}_i(\omega)$.

B. Suppose that $s_1(t)$ and $s_2(t)$ are Gaussian noises, whose power spectra are $P_1(\omega)$ and $P_2(\omega)$.
Assuming they are independent, calculate the power spectra of $r_1(t)$ and $r_2(t)$, and the cross-spectrum of $r_1(t)$ and $r_2(t)$.

C. As in B, but now assume that $s_1(t)$ and $s_2(t)$ have a nonzero cross-spectrum $X(\omega)$. 
Q4. Graph Laplacians of composite graphs (2 parts, 1 point for each part)

A. Say \( G \) is a connected graph of size \( n_G \) whose graph Laplacian \( L_G \) has eigenvectors \( \varphi_i \) with eigenvalues \( \lambda_i \), and \( H \) is a connected graph of size \( n_H \) whose graph Laplacian \( L_H \) has eigenvectors \( \psi_i \) with eigenvalues \( \mu_i \). For definiteness, take \( \varphi_1 \) to be the uniform eigenvector, i.e., the eigenvector composed of all 1’s, for which \( L_G \varphi_1 = 0 \), and, similarly, for \( \psi_1 \).

Consider the graph \( K \) of size \( n_G + n_H \) consisting of all the vertices and edges in \( G \) and \( H \), along with an edge from every vertex in \( G \) to every vertex in \( H \). Find the eigenvectors and eigenvalues of the graph Laplacian of \( K \).

![Graph K](image)

B. Say \( G \) is a connected graph of size \( n_G \) whose graph Laplacian \( L_G \) has eigenvectors \( \varphi_i \) with eigenvalues \( \lambda_i \), and \( H \) is a connected graph of size \( n_H \) whose graph Laplacian \( L_H \) has eigenvectors \( \psi_i \) with eigenvalues \( \mu_i \). For definiteness, take \( \varphi_1 \) to be the uniform eigenvector, i.e., the eigenvector composed of all 1’s, for which \( L_G \varphi_1 = 0 \), and, similarly, for \( \psi_1 \).

Consider the graph \( B \) of size \( n_G + n_H + 1 \) consisting of all the vertices and edges in \( G \) and \( H \), along with a new vertex \( P \). There are edges from every vertex in \( G \) to \( P \), and from every vertex in \( H \) to \( P \). (If \( P \) is positioned between \( G \) and \( H \), \( B \) is a “bowtie”).

Find the eigenvectors and eigenvalues of the graph Laplacian of \( B \).

![Graph B](image)
Q5. Graph Laplacians of bilaterally symmetric graphs (2 parts, 1 point for each part)

Say $G$ is a connected graph of size $n_G$ whose graph Laplacian $L_G$ has eigenvectors $\varphi_i$ with eigenvalues $\lambda_i$. For definiteness, take $\varphi_1$ to be the uniform eigenvector, i.e., the eigenvector composed of all 1’s, for which $L_G\varphi_1 = 0$.

Now form a graph $Y$ that consists of two copies of $G$, and, between these two copies, the corresponding nodes are connected. (Think of $G$ as being the graph that represents connections within a hemisphere, and $Y$ as being the graph that represents the two hemispheres, with their internal connections and callosal connections between the corresponding areas.)

There’s an obvious two-element group $R = \{e, r\}$ that leaves $Y$ invariant: the non-identity element of $R$ interchanges the two components of $Y$.

A. What does the action of $R$ on $Y$ imply about the eigenvectors of the graph Laplacian $L_Y$?

B. Determine the eigenvectors and eigenvalues of the graph Laplacian $L_Y$. 
Q6: Community structure (2 parts, 1 point for each part)

Recall that the community structure of a graph consists of an assignment of each vertex $i$ to a community $c_i$ that maximizes the “quality function”

$$Q = \sum_{i,j} (a_{ij} - p_{ij}) \delta(c_i, c_j),$$

where the sum is over all distinct pairs of vertices $\{i, j\}$, $p_{ij} = \frac{d_i d_j}{2m}$, where $m$ is the total number of edges, and $d_i$ is the degree of the vertex $i$.

A. For a cyclic graph $G$ of size $n \geq 3$, find the community structure.
B. For a linear graph $G$ of size $n \geq 3$, find the community structure.