Groups, Fields, and Vector Spaces

Homework #1 (2014-2015)

Q1: Group or not a group?

Which of the following are groups? If a group, is it commutative? Finite or infinite? If infinite, is it discrete or continuous? If not a group, where does it fail?

A. The even integers $\{... -6, -4, -2, 0, 2, 4, 6...\}$, under multiplication

B. The set of all translations of 3-space, under composition

C. The set of all rotations of 3-space, under composition

D. The set of all $N \times N$ matrices with integer entries, under matrix addition

E. The set of all $N \times N$ matrices with integer entries, under matrix multiplication

F. The set of all 2×2 matrices with integer entries and determinant 1, under matrix multiplication

Q2. Modular arithmetic

For two integers *x* and *y*, we say $x = y$ (mod *k*) if *x* and *y* differ by an integer multiple of *k*. So, for example, 3+4=2 (mod 5) and 6*9=10 (mod 11).

A. Show that the integers $\{0, 1, \ldots k-1\}$ form a group under addition (mod *k*).

B. For what integers *k* do the integers $\{1, \ldots k-1\}$ form a group under multiplication (mod *k*)?

Q3. Normal subgroups

Definition: A subgroup *H* of *G* is said to be a "normal" subgroup if, for any element *g* of *G* and any element *h* of *H*, the combination ghg^{-1} is also a member of *H*.

A. Show that if φ is a homomorphism from *G* to some other group *R*, then the kernel of φ is a normal subgroup of *G*. (In class, showed that the kernel must be a subgroup, here, show that it is normal as well.)

B. Show that if *H* is a normal subgroup and *b* is any element of *G*, then the right coset *Hb* is equal to the left coset, *bH*.

C. Show that if H is a normal subgroup, then any element of the right coset Hb, composed with any element of the right coset Hc, is a member of the right coset Hbc, with the product bc carried out according to the group operation in G.

D. Consider the mapping from group elements to cosets, $\varphi(b) = Hb$ (where H is a normal subgroup). Show that this constitutes a homomorphism from the group G to the set of cosets, with the group operation on cosets defined by $(Hb) \circ (Hc) = Hbc$.

E. Find the kernel of the homomorphism in D.