Q1: Impulse responses and transfer functions

A. Exponential decay: For a system $F$ with an impulse response $f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$, find the transfer function $\hat{f}(\omega)$.

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt = \int_{0}^{\infty} e^{-i\omega t} \lambda e^{-\lambda t} dt = \lambda \int_{0}^{\infty} e^{-(i\omega + \lambda)t} dt$$

$$= -\left( i\omega + \lambda \right) e^{-(i\omega + \lambda)} \frac{\lambda}{i\omega + \lambda} = -\frac{1}{1 + i\omega / \lambda}$$

B. Pure delay: For a system $F_T$ with an impulse response $f_T(t) = \delta(t - T)$, find the transfer function $\hat{f}_T(\omega)$.

$$\hat{f}_T(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f_T(t) dt = \int_{0}^{\infty} e^{-i\omega t} \delta(t - T) dt = e^{-i\omega T}$$

C. Differentiation: Consider a system $F_{diff}$ whose output is the derivative of the input. We can’t write an impulse response for this system in a straightforward way, because the derivative of a delta-function is not defined. But we can determine its transfer function, by considering its response to sinusoids $e^{i\omega t}$. What is its transfer function $\hat{f}_{diff}(\omega)$?

$F_{diff}$ takes $e^{i\omega t}$ into $\frac{d}{dt} e^{i\omega t} = i\omega e^{i\omega t}$. So $F_{diff}$ multiplies $e^{i\omega t}$ by $i\omega$. Therefore, $\hat{f}_{diff}(\omega) = i\omega$.

Note that $F_{diff} = \lim_{T \to 0} \frac{f'_0 - f'_1}{T}$ (the definition of the derivative). Correspondingly,

$$\lim_{T \to 0} \frac{\hat{f}_0(\omega) - \hat{f}_1(\omega)}{T} = \lim_{T \to 0} \frac{1}{T} e^{-i\omega T} = \lim_{T \to 0} \frac{1 - (1 - i\omega T)}{T} = \lim_{T \to 0} \frac{i\omega T}{T} = i\omega = \hat{f}_{diff}(\omega)$$

Q2: Biased diffusion

In the notes, we modeled diffusion as a random walk from $x = 0$ to $x = \pm b$, with equal probability, in time $\Delta T$. That is, $F_{\Delta T}(x) = \frac{1}{2} (\delta(x - b) + \delta(x + b))$. We saw that this had a stable limit as $\Delta T \to 0$ if $b^2 = A\Delta T$, i.e., $b = \sqrt{A\Delta T}$.

Now consider a biased process, in which the probability of a step to $+b$ is $\frac{1}{2} (1 + \alpha)$ and the probability of a step to $-b$ is $\frac{1}{2} (1 - \alpha)$. So now, $F_{\Delta T}(x) = \frac{1}{2} ((1 + \alpha)\delta(x - b) + (1 - \alpha)\delta(x + b))$. 
A. Determine $\hat{F}_{\Delta T}(\omega)$.

$$\hat{F}_{\Delta T}(\omega) = \int_{-\infty}^{\infty} F_{\Delta T}(x)e^{-i\omega x}dx = \frac{1}{2} \int_{-\infty}^{\infty} ((1+\alpha)\delta(x-b)+(1+\alpha)\delta(x+b))e^{-i\omega x}dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} (\delta(x-b)+\delta(x+b))e^{-i\omega x}dx + \frac{\alpha}{2} \int_{-\infty}^{\infty} (\delta(x-b)-\delta(x+b))e^{-i\omega x}dx$$

$$= \frac{1}{2}(e^{-i\omega b} + e^{i\omega b}) + \frac{\alpha}{2}(e^{-i\omega b} - e^{i\omega b}) = \cos(\omega b) - i\alpha \sin(\omega b)$$

B. How should $\alpha$ vary with $\Delta T$ to ensure a stable limit for $\hat{F}_T(\omega) = \lim_{\Delta T \to 0} \left( \hat{F}_{\Delta T}(\omega) \right)^{T/\Delta T}$ as $\Delta T \to 0$, and what is this limit?

For small $b$

$$\hat{F}_{\Delta T}(\omega) = \cos(\omega b) - i\alpha \sin(\omega b) \approx 1 - \frac{1}{2}b^2\omega^2 - i\alpha wb \approx e^{-\frac{1}{2}b^2\omega^2 - i\alpha wb}.$$  

So

$$\left( \hat{F}_{\Delta T}(\omega) \right)^{T/\Delta T} \approx e^{-\frac{1}{2}b^2\omega^2 - i\alpha wb \frac{T}{\Delta T}}.$$  

We need $b^2/\Delta T$ and $\alpha b / \Delta T$ to have a stable limit as $\Delta T \to 0$. So $b^2 / \Delta T = A$ implies $b = \sqrt{A\Delta T}$, and $\alpha b / \Delta T = h$ implies $\alpha = h\Delta T / b = \frac{h\Delta T}{\sqrt{A\Delta T}} = h\sqrt{\frac{\Delta T}{A}}$.

So

$$\hat{F}_T(\omega) = \lim_{\Delta T \to 0} \left( \hat{F}_{\Delta T}(\omega) \right)^{T/\Delta T} = e^{-\frac{1}{2}b^2\omega^2 - i\alpha wb \frac{T}{\Delta T}} = e^{-\omega^2/2 - i\alpha wbT/\Delta T} = e^{-\omega^2AT/2 - i\alpha wbT}.$$

C. If, at time 0, the distribution is $p_0(x) = \delta(x)$, what is the distribution $p_T(x)$ at time $T$?

$$\hat{p}_T(\omega) = \hat{F}_T(\omega)\hat{p}(0),$$  

and since $p_0(x) = \delta(x)$, $\hat{p}(0) = 1$. So $\hat{p}_T(\omega) = \hat{F}_T(\omega) = e^{-\omega^2AT/2 + i\alpha wbT}. Inverting the transform,

$$p_T(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}_T(\omega)e^{i\omega x}d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2AT/2 - i\alpha wbT}e^{i\omega x}d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2AT/2}e^{i\omega(x-hT)}d\omega.$$  

The final integral is the same as the one that arose for unbiased diffusion (notes), but with $x - hT$ replacing $x$.

So $p_T(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}_T(\omega)e^{i\omega x}d\omega = \frac{1}{\sqrt{2\pi AT}} e^{-(x-hT)^2/2AT}$, a Gaussian centered at $x = hT$ whose variance is $\sqrt{AT}$.

Not surprisingly, when the probabilities of leftward and rightward steps are unequal, the distribution drifts by an amount proportional to time.