Linear Systems, Black Boxes, and Beyond

Homework #2 (2014-2015), Answers

Q1: Power spectra of some random processes.

Say an output, \( y(t) \), is related to an input, \( x(t) \) by \( \frac{dy}{dt} = x - ky \). That is, \( y \) integrates \( x \), but tends to return (decay) to 0 at a rate \( k \).

A. Determine the transfer function that relates \( y \) to \( x \). (Hint – find the response to \( x(t) = e^{i\omega t} \)).

The relationship between \( y(t) \) and \( x(t) \) is linear and time-translation-invariant. So if \( x(t) = e^{i\omega t} \), then \( y(t) \) must be a multiple of \( x(t) \). So say \( y(t) = \hat{L}(\omega)e^{i\omega t} \). Then \( \frac{dy}{dt} = i\omega \hat{L}(\omega)e^{i\omega t} \).

Substituting for each of the terms in \( \frac{dy}{dt} = x - ky \), \( i\omega \hat{L}(\omega)e^{i\omega t} = e^{i\omega t} - k\hat{L}(\omega)e^{i\omega t} \). Solving for \( \hat{L}(\omega) \) yields \( (i\omega + k)\hat{L}(\omega)e^{i\omega t} = e^{i\omega t} \), or, \( \hat{L}(\omega) = \frac{1}{k + i\omega} \).

Note the resemblance to Q1 of last week.

B. In the above scenario, if \( x(t) \) is white noise with power per bandwidth equal to \( a \), i.e., \( P_x(\omega) = a \), find \( P_y(\omega) \).

\[
P_y(\omega) = \left| \hat{L}(\omega) \right|^2 P_x(\omega), \text{ so } P_y(\omega) = \left| \frac{1}{k + i\omega} \right|^2 a = \frac{a}{k^2 + \omega^2}.
\]

C. In the limit that the rate is extremely slow (i.e., as \( k \to 0 \)), the above system simply integrates its input. What is its power spectrum?

\[
P_y(\omega) = \lim_{k \to 0} \frac{a}{k^2 + \omega^2} = \frac{a}{\omega^2}.
\]

Q2. Say a system \( F \) is a parallel combination of two systems: one component is \( 2kL \) (where \( L \) is as above); the second is system whose response to \( x(t) \) is \( -x(t) \).

A. What is the transfer function \( \hat{F}(\omega) \)?

Since \( F = 2kL - I \), \( \hat{F}(\omega) = 2k\hat{L}(\omega) - 1 = \frac{2k}{k + i\omega} - 1 = \frac{k - i\omega}{k + i\omega} \).

B. Given an input \( x(t) \) and an output \( y(t) \), how are the power spectra of input and output related?

\[
P_y(\omega) = \left| \hat{F}(\omega) \right|^2 P_x(\omega) = \left( \frac{\left| k - i\omega \right|^2}{k + i\omega} \right)^2 P_x(\omega) = \left( \frac{\left| k - i\omega \right|^2}{k + i\omega} \right)^2 P_x(\omega) = P_x(\omega).
\]

Note that this shows that knowing that the amplitude of a transfer function is unity does not mean that the transfer function itself is unity, or even just a delay \( e^{-i\omega t} \) – in this case, \( \left| \hat{F}(\omega) \right| = 1 \) but \( \hat{F}(\omega) = \frac{k - i\omega}{k + i\omega} \).