Linear Systems, Black Boxes, and Beyond

Homework #2 (2014-2015), Answers

Q1: Power spectra of some random processes.

Say an output, $y(t)$*, is related to an input,* $x(t)$ *by* $\frac{dy}{dt} = x - ky$*. That is,* y *integrates* x*, but tends to return (decay) to 0 at a rate* k*.*

A. Determine the transfer function that relates y *to* x*. (Hint – find the response to* $x(t) = e^{i\omega t}$*).*

The relationship between $y(t)$ and $x(t)$ is linear and time-translation-invariant. So if $x(t) = e^{i\omega t}$, then $y(t)$ must be a multiple of $x(t)$. So say $y(t) = \hat{L}(\omega)e^{i\omega t}$. Then $\frac{dy}{dt} = i\omega\hat{L}(\omega)e^{i\omega t}$.
Substituting for each of the terms in $\frac{dy}{dt} = x - ky$, $i\omega\hat{L}(\omega)e^{i\omega t} = e^{i\omega t} - k\hat{L}(\omega)e^{i\omega t}$. Solving for $\hat{L}(\omega)$ yields $(i\omega + k)\hat{L}(\omega)e^{i\omega t} = e^{i\omega t}$, or, $\hat{L}(\omega) = \frac{1}{k + i\omega}$.

Note the resemblance to Q1 of last week.

B. In the above scenario, if $x(t)$ *is white noise with power per bandwidth equal to* a*, i.e.,* $P_X(\omega) = a$*, find* $P_Y(\omega)$*.*

$P_Y(\omega) = \left|\hat{L}(\omega)\right|^2 P_X(\omega)$, so $P_Y(\omega) = \left|\frac{1}{k + i\omega}\right|^2 a = \frac{a}{k^2 + \omega^2}$.

C. In the limit that the rate is extremely slow (i.e., as $k \to 0$*), the above system simply integrates its input. What is its power spectrum?*

$P_Y(\omega) = \lim_{k \to 0} \frac{a}{k^2 + \omega^2} = \frac{a}{\omega^2}$.

Q2. Say a system F *is a parallel combination of two systems: one component is* $2kL$ *(where* L *is as above); the second is system whose response to* $x(t)$ *is* $-x(t)$*.*

A. What is the transfer function $\hat{F}(\omega)$*?*

Since $F = 2kL - I$, $\hat{F}(\omega) = 2k\hat{L}(\omega) - 1 = \frac{2k}{k + i\omega} - 1 = \frac{k - i\omega}{k + i\omega}$.

B. Given an input $x(t)$ *and an output* $y(t)$*, how are the power spectra of input and output related?*

$P_Y(\omega) = \left|\hat{F}(\omega)\right|^2 P_X(\omega) = \left|\frac{k - i\omega}{k + i\omega}\right|^2 P_X(\omega) = \frac{\left|k - i\omega\right|^2}{\left|k + i\omega\right|^2} P_X(\omega) = P_X(\omega)$.

Note that this shows that knowing that the amplitude of a transfer function is unity does not mean that the transfer function itself is unity, or even just a delay $e^{-i\omega T}$ – in this case, $\left|\hat{F}(\omega)\right| = 1$ but $\hat{F}(\omega) = \frac{k - i\omega}{k + i\omega}$.