Linear Transformations and Group Representations

Homework #1 (2014-2015), Answers

Q1: Eigenvectors of some linear operators (linear transformations) in matrix form

In each case, use the characteristic equation to find the eigenvalues, the eigenvectors, the dimensions of the eigenspaces, and whether a basis can be chosen from the eigenvectors.

A. $Q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, *with* $p \neq q$.

First, use the characteristic equation to find the eigenvalues.

$\det(zI - Q) = \det\begin{pmatrix} z-p & 0 \\ 0 & z-q \end{pmatrix} = (z-p)(z-q)$. $\det(zI-Q) = 0$ requires $(z-p)(z-q) = 0$, i.e., $z = p$ or $z = q$.

To find the eigenvector with eigenvalue p (if you can't guess what it is): Say V has basis elements e_1 and e_2, expressed as columns $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the eigenvector is $v = ae_1 + be_2 = \begin{pmatrix} a \\ b \end{pmatrix}$. Then $Qv = Q\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} pa \\ qb \end{pmatrix} = pae_1 + qbe_2$. So $Qv = pv$ implies that $Qv = pae_1 + qbe_2$ and $pv = pae_1 + pbe_2$ are the same, which means that $pae_1 + qbe_2 = pae_1 + pbe_2$, i.e., $qbe_2 = pbe_2$. Since $p \neq q$, this means that $b = 0$, i.e., $v = ae_1$.
By the same argument, the eigenvector for eigenvalue q is $v = be_2$.
The eigenvectors are each multiples of the coordinate basis, and therefore form a basis.

In other words, the eigenvalues of a diagonal matrix are the entries on the diagonal, and the eigenvector corresponding to the kth element on the diagonal is a vector along the kth coordinate.

B. $A = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, *with* $r \neq 0$.

First, use the characteristic equation to find the eigenvalues.

$\det(zI - A) = \det\begin{pmatrix} z-1 & -r \\ 0 & z-1 \end{pmatrix} = (z-1)^2$.

$\det(zI - A) = 0$ requires $z = 1$, so the only eigenvalue of A is 1.

As in part A, say V has basis elements e_1 and e_2, expressed as columns $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Then $Ae_1 = e_1$ and $Ae_2 = re_1 + e_2$. So e_1 is an eigenvector of eigenvalue 1. To look for any others: Let $v = ae_1 + be_2$. Then $Av = A(ae_1 + be_2) = ae_1 + b(re_1 + e_2) = (a + br)e_1 + be_2$.

$Av = v$ implies $ae_1 + be_2 = (a+br)e_1 + be_2$. Since e_1 and e_2 are linearly independent (they form a basis), their coefficients must be equal. For e_1, this requires $a = a + br$, i.e., $b = 0$. For e_2, the coefficients are always equal. So the only eigenvalues have $b = 0$, i.e., the only eigenvalues are e_1 and its multiples.

So there is one eigenvalue, 1, whose eigenspace has dimension 1, spanned by the eigenvector $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since A operates in a two-dimensional vector space, the eigenvectors cannot form a basis.

C. $B = \begin{pmatrix} q & r \\ r & q \end{pmatrix}$ *(assume* $q > r > 0$*).*

Again, first use the determinant to find the eigenvalues.

$\det(zI - B) = \det \begin{pmatrix} z-q & -r \\ -r & z-q \end{pmatrix} = (z-q)^2 - r^2$. $\det(zI - B) = 0$ solves for $z = q \pm r$, so these are the eigenvalues of B. To find the eigenvectors: As in part A, say $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and v is an eigenvector with $v = ae_1 + be_2$. $Be_1 = qe_1 + re_2$. $Be_2 = re_1 + qe_2$. So

$Bv = aBe_1 + bBe_2 = a(qe_1 + re_2) + b(re_1 + qe_2) = (aq + br)e_1 + (ar + bq)e_2$.

Looking for the eigenvector of eigenvalue $q + r$:

$Bv = (q+r)v$ implies $(aq+br)e_1 + (ar+bq)e_2 = (q+r)ae_1 + (q+r)be_2$. Since e_1 and e_2 are linearly independent, equality can only hold if coefficients of e_1 match, and coefficients of e_2 match.

For e_1: $aq + br = aq + ar$; For e_2: $ar + bq = bq + br$. Both solve for $a = b$. So the eigenvectors corresponding to the eigenvalue $q + r$ are multiples of $e_1 + e_2$, i.e., of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For the eigenvectors of eigenvalue $q - r$:

$Bv = (q-r)v$ implies $(aq+br)e_1 + (ar+bq)e_2 = (q-r)ae_1 + (q-r)be_2$. Since e_1 and e_2 are linearly independent, equality can only hold if coefficients of e_1 match, and coefficients of e_2 match.

For e_1: $aq + br = aq - ar$; For e_2: $ar + bq = bq - br$. Both solve for $a = -b$. So the eigenvectors corresponding to the eigenvalue $q - r$ are multiples of $e_1 - e_2$, i.e., of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

So there are two eigenvalues, $q + r$ and $q - r$, each with eigenspace of dimension 1, spanned by $e_1 + e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $e_1 - e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. They form a basis.

D. $C = \begin{pmatrix} q & r \\ -r & q \end{pmatrix}$

Again, first use the determinant to find the eigenvalues.

$\det(zI - B) = \det\begin{pmatrix} z-q & -r \\ r & z-q \end{pmatrix} = (z-q)^2 + r^2$. $\det(zI - B) = 0$ solves for $z = q \pm ir$, so these are the eigenvalues of *B*. To find the eigenvectors: As in part *B*, say $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and *v* is an eigenvector with $v = ae_1 + be_2$. $Ce_1 = qe_1 + re_2$. $Ce_2 = -re_1 + qe_2$. So $Cv = aCe_1 + bCe_2 = a(qe_1 + re_2) + b(-re_1 + qe_2) = (aq - br)e_1 + (ar + bq)e_2$.

Looking for the eigenvector of eigenvalue $q + ir$:

$Cv = (q+ir)v$ implies $(aq - br)e_1 + (ar + bq)e_2 = (q+ir)ae_1 + (q+ir)be_2$. Since e_1 and e_2 are linearly independent, equality can only hold if coefficients of e_1 match, and coefficients of e_2 match.

For e_1: $aq - br = aq + air$; For e_2: $ar + bq = bq + bir$. Both solve for $a = ib$. So the eigenvectors corresponding to the eigenvalue $q + ir$ are multiples of $e_1 + ie_2$, i.e., of $\begin{pmatrix} 1 \\ i \end{pmatrix}$.

Similarly (or, remembering that everything is symmetric with respect to complex conjugation) the eigenvector associated with $z = q - ir = \overline{q + ir}$ must be $\overline{\begin{pmatrix} 1 \\ i \end{pmatrix}} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Q2: Eigenvectors of derived linear operators

Say X and Y are linear transformations from a vector space V to itself, and v is an eigenvector both of X, with eigenvalue λ_X *, and of Y, with eigenvalue* λ_Y*.*

A. Show that v is also an eigenvector of the transformation $X + Y$ *, and find its eigenvalue.*

$(X+Y)v = Xv + Yv = \lambda_X v + \lambda_Y v = (\lambda_X + \lambda_Y)v$. So the eigenvalue is $\lambda_X + \lambda_Y$.

B. Show that v is also an eigenvector of the transformation αX *,where* α *is a scalar, and find its eigenvalue.*

$(\alpha X)v = \alpha(Xv) = \alpha(\lambda_X v) = (\alpha\lambda_X)v$. So the eigenvalue is $\alpha\lambda_X$.

C. Show that v is also an eigenvector of the transformation XY *, and find its eigenvalue.*

$(XY)v = X(Yv) = X(\lambda_Y v) = \lambda_Y X(v) = \lambda_Y \lambda_X v = (\lambda_X \lambda_Y)v$. So the eigenvalue is $\lambda_X \lambda_Y$.

D. Show that v is also an eigenvector of the transformation $XY - YX$ *, and find its eigenvalue.*

Part C says that v is an eigenvector of XY, with eigenvalue $\lambda_X \lambda_Y$. Part C also says that v is an eigenvector of YX, with eigenvalue $\lambda_Y \lambda_X$. So part B (with $\alpha = -1$) says that v is an eigenvector of $-YX$, with eigenvalue $-\lambda_Y \lambda_X$. Now apply part A to the sum of two transformations, XY and $-YX$. v is an eigenvector of each, so it is an eigenvector of their sum, $XY - YX$. Ant the eigenvalue is the sum of the component eigenvalues, $\lambda_X \lambda_Y - \lambda_Y \lambda_X = 0$. Interesting –since X and Y are matrices, they typically don't commute, i.e., we can't expect that $XY - YX = 0$. But if they share a common eigenvector, it is an eigenvector that has an eigenvalue of 0.