Linear Transformations and Group Representations

Homework #4 (2014-2015), Answers

Q1. Irreducible representations of the dihedral group of D_4

Here we build on the homework from last week to construct all the irreducible representations of dihedral group D_4, i.e., the rotations and reflections of a square. We continue to use the following notation for its elements:

I *: the identity*
R, R^{-1} *: 90-degree rotations right and left (* $R^4 = I$ *)*
C *: rotation by 180 deg (* $C^2 = I$, $R^2 = C$ *)*
X , Y *: mirror flips in the x- and y-axes (* $X^2 = Y^2 = 1$*)*
$M_\backslash$, $M_/$ *: mirror flips on the two diagonals (* $M_\backslash^2 = M_/^2 = I$ *).*

We had the following table of characters – the last line added in class and is the representation that maps a group element to +1 or -1 depending on whether it exchanges the front and back faces of the square:

L	I	R, R^{-1}	C	X , Y	$M_\backslash$, $M_/$
L_I : *Trivial rep*	*1*	*1*	*1*	*1*	*1*
$L_{2\times 2}$: 2×2 *matrices*	*2*	*0*	*-2*	*0*	*0*
L_{corner} : *Corner perm*	*4*	*0*	*0*	*0*	*2*
L_{side} : *Side perm*	*4*	*0*	*0*	*2*	*0*
L_{diag} : *Diag perm*	*2*	*0*	*2*	*0*	*2*
L_{face} : *Face exchange*	*1*	*1*	*1*	*-1*	*-1*

We also determined that L_{corner}, L_{side}, *and* L_{diag} *contained one copy of the trivial representation, since for each of those,* $\frac{1}{|G|}\sum_g \chi_L(g) = 1$.

A. For the representations L *that contain the trivial representation, replace their entries in the above character table with the characters of the smaller representations* L' *for which* $L = L' \oplus L_I$.

Since the character of a representation M at a group element g is given by $\chi_M(g) = tr(M_g)$, then $\chi_L(g) = tr(L_g) = tr(L'_g \oplus I_g) = tr(L'_g) + 1 = \chi_{L'}(g) + 1$, so $\chi_{L'}(g) = \chi_L(g) - 1$.

L	I	R, R^{-1}	C	X , Y	$M_\backslash$, $M_/$
L_I : Trivial rep	1	1	1	1	1
$L_{2\times 2}$: 2×2 matrices	2	0	-2	0	0
L'_{corner} : Corner perm	3	-1	-1	-1	1

L'_{side} : Side perm	3	-1	-1	1	-1
$L_{diag}{}'$: Diag perm	1	-1	1	-1	1
L_{face} : Face exchange	1	1	1	-1	-1

B. Using the Group Representation Theorem characterization that for irreducible representations, characters are orthonormal, identify the representations that are reducible.

For an irreducible representation, $d(L,L)=\frac{1}{|G|}\sum_g |\chi_L(g)|^2 = 1$. So we calculate:

$d(L_{2\times 2},L_{2\times 2})=\frac{1}{8}\left(2^2+2\bullet(0)^2+(-2)^2+2\bullet(0)^2+2\bullet(0)^2\right)=1$, therefore irreducible

$d(L'_{corner},L'_{corner})=\frac{1}{8}\left(3^2+2\bullet(-1)^2+(-1)^2+2\bullet(-1)^2+2\bullet(1)^2\right)=2$, therefore reducible

$d(L'_{side},L'_{side})=\frac{1}{8}\left(3^2+2\bullet(-1)^2+(-1)^2+2\bullet(1)^2+2\bullet(-1)^2\right)=2$, therefore reducible

$d(L'_{diag},L'_{diag})=\frac{1}{8}\left(1^2+2\bullet(-1)^2+(1)^2+2\bullet(-1)^2+2\bullet(1)^2\right)=1$, therefore reducible (had to be, dim= 1)

$d(L'_{face},L'_{face})=\frac{1}{8}\left(1^2+2\bullet(1)^2+(1)^2+2\bullet(-1)^2+2\bullet(-1)^2\right)=1$, therefore reducible (had to be, dim= 1)

L	I	R , R^{-1}	C	X , Y	$M_\backslash$, $M_/$	
L_I : Trivial rep	1	1	1	1	1	irreducible
$L_{2\times 2}$: 2×2 matrices	2	0	-2	0	0	irreducible
L'_{corner} : Corner perm	3	-1	-1	-1	1	reducible
L'_{side} : Side perm	3	-1	-1	1	-1	reducible
$L_{diag}{}'$: Diag perm	1	-1	1	-1	1	irreducible
L_{face} : Face exchange	1	1	1	-1	-1	irreducible

C. Recall that $d(L,M)=\frac{1}{|G|}\sum_g \overline{\chi_L(g)}\chi_M(g)$ indicates how ways that an irreducible piece of a representation L can be matched to an irreducible piece of a representation M. So if L is irreducible, it indicates how many copies of L are inside of M. Use this to further reduce the remaining reducible representations.

First, let's see what's inside of L'_{corner} ($d(L,M)=\frac{1}{|G|}\sum_g \overline{\chi_L(g)}\chi_M(g)$ with $M=L'_{corner}$, and L being one of the known-irreducible representations):

$d(L_{2\times 2},L'_{corner})=\frac{1}{8}\left(2\bullet 3+2\bullet(0\bullet(-1))+(-2)\bullet(-1)+2\bullet(0\bullet(-1))+2\bullet(0\bullet(1))\right)=\frac{8}{8}=1$

$d(L'_{diag},L'_{corner})=\frac{1}{8}\left(1\bullet 3+2\bullet((-1)\bullet(-1))+(1)\bullet(-1)+2\bullet((-1)\bullet(-1))+2\bullet(1\bullet 1)\right)=\frac{8}{8}=1$

$$d(L_{face}, L'_{corner}) = \frac{1}{8}\left(1\bullet 3 + 2\bullet((1)\bullet(-1)) + (1)\bullet(-1) + 2\bullet((-1)\bullet(-1)) + 2\bullet((-1)\bullet 1)\right) = \frac{0}{8} = 0$$

So we've accounted for all of the three dimensions of L'_{corner}: $L'_{corner} = L_{2\times 2} \oplus L'_{diag}$, and one chan check that $\chi_{L'_{corner}}(g) = \chi_{L_{2\times 2}}(g) + \chi_{L'_{diag}}(g)$. So we don't find anything new inside of L'_{corner}.

Next, let's see what's inside of L'_{side} ($d(L,M) = \frac{1}{|G|}\sum_g \overline{\chi_L(g)}\chi_M(g)$ with $M = L'_{side}$, and L being one of the known-irreducible representations):

$$d(L_{2\times 2}, L'_{side}) = \frac{1}{8}\left(2\bullet 3 + 2\bullet(0\bullet(-1)) + (-2)\bullet(-1) + 2\bullet(0\bullet(1)) + 2\bullet(0\bullet(-1))\right) = \frac{8}{8} = 1$$

$$d(L'_{diag}, L'_{side}) = \frac{1}{8}\left(1\bullet 3 + 2\bullet((-1)\bullet(-1)) + (1)\bullet(-1) + 2\bullet((-1)\bullet(1)) + 2\bullet(1\bullet(-1))\right) = \frac{0}{8} = 0$$

$$d(L_{face}, L'_{corner}) = \frac{1}{8}\left(1\bullet 3 + 2\bullet((1)\bullet(-1)) + (1)\bullet(-1) + 2\bullet((-1)\bullet(1)) + 2\bullet((-1)\bullet(-1))\right) = \frac{0}{8} = 0$$

So we've accounted for only two of the three dimensions of L'_{side}. There's one more dimension. That is, $L'_{side} = L_{2\times 2} \oplus L_X$, where $\chi_{L'_{side}}(g) = \chi_{L_{2\times 2}}(g) + \chi_{L_X}(g)$. So $\chi_{L_X}(g) = \chi_{L'_{side}}(g) - \chi_{L_{2\times 2}}(g))$, and we will add this new irreducible representation to the character table.

L	I	R, R^{-1}	C	X , Y	$M_\backslash$, $M_/$	
L_I: Trivial rep	1	1	1	1	1	irreducible
$L_{2\times 2}$: 2×2 matrices	2	0	-2	0	0	irreducible
$L_{diag}{}'$: Diag perm	1	-1	1	-1	1	irreducible
L_{face}: Face exchange	1	1	1	-1	-1	irreducible
L_X: $L'_{side} = L_{2\times 2} \oplus L_X$	1	-1	1	1	-1	irreducible

D. Show that the table now has all of the irreducible representations of the dihedral group.

Method 1: There are five conjugate classes (the columns), and the five rows are all different. Since the irreducible characters are orthonormal functions on the conjugate classes, there can't be any more.

Method 2: The regular representation contains each irreducible representation, with a multiplicity equal to its dimension. That is, if we have all the irreducible representations, then $\chi_{regular} = \chi_{L_I} + 2\chi_{L_{2\times 2}} + \chi_{L'_{diag}} + \chi_{L_{face}} + \chi_{L_X}$. This can be directly verified by summing along the columns of the character table (note the factor of 2 for the 2-dimensional representation $L_{2\times 2}$).

Q2. Representations of subgroups: an irreducible representation may become reducible, when restricted to a subgroup.

Setup: A representation L of a group G is, necessarily, a representation for any subgroup H of G, simply by restricting it to $g \in H$. But if a representation is irreducible on G, it need not be irreducible *on H. A trivial example of this is to start with a representation of dimension $d > 1$, and restrict it to the one-element identity subgroup of G; in this case, the representation maps the identity element to the $d \times d$ identity matrix – which clearly is reducible. But here's a less-trivial example that illustrates what is more generic.*

We consider the cyclic group $\mathbb{Z}_4$, which is the rotation group of the square – and hence, a subgroup of D_4 considered in Q1.As in the class notes, $\mathbb{Z}_n$ it has a representation L_m for every nth root of unity, which takes a $2\pi/n$ rotation to $\exp(\frac{2\pi i}{n} m)$. Here $n = 4$, and we adopt the notation of Q1, so R is a rotation by $\pi/2$, $R^{-1} = R^3$ is a rotation by $3\pi/2$, and $C = R^2$ is a rotation by π. So the character table of $\mathbb{Z}_n$ is

L	I	R	$R^2 = C$	$R^3 = R^{-1}$
L_0 $(m = 0)$	1	1	1	1
L_1 $(m = 1)$	1	*i*	-1	-*i*
L_2 $(m = 2)$	1	-1	1	-1
L_3 $(m = 3)$	1	-*i*	-1	*i*

Now consider the irreducible representations of D_4, determined in Q1. Find their characters, considered as a representation of $\mathbb{Z}_4$. Which ones are reducible, and which are irreducible? How do they relate to the above irreducible representations of $\mathbb{Z}_4$?

The characters of the representations are just the characters on D_4 restricted to $\mathbb{Z}_4$. So the table is:

L	I	R	C	R^{-1}
L_I : Trivial rep	1	1	1	1
$L_{2\times 2}$: 2×2 matrices	2	0	-2	0
$L_{diag}{}'$: Diag perm	1	-1	1	-1
L_{face} : Face exchange	1	1	1	1
L_X : $L'_{side} = L_{2\times 2} \oplus L_X$	1	-1	1	-1

The one-dimensional representations have to be irreducible. Comparing characters of the two tables: $L_I = L_0$, $L_{diag}{}' = L_2$, $L_{face} = L_0$, $L_X = L_2$.

The two-dimensional representation $L_{2\times 2}$ must be reducible, since $\mathbb{Z}_4$ is commutative. Also, by the trace formula, $d(L_{2\times 2}, L_{2\times 2}) = \frac{1}{|G|}\sum_g \left|\chi_{L_{2\times 2}}(g)\right|^2 = \frac{1}{4}\left(2^2 + (-2)^2\right) = 2$, confirming that $L_{2\times 2}$ is reducible.

We can find its components by projecting the character of $L_{2\times 2}$ on the irreducible representations of $\mathbb{Z}_4$:

$$d(L_{2\times 2}, L_0) = d(L_{2\times 2}, L_2) = \frac{1}{4}\left(2\bullet 1 + (-2)\bullet 1\right) = 0\,,$$

$$d(L_{2\times 2}, L_1) = d(L_{2\times 2}, L_3) = \frac{1}{4}\left(2\bullet 1 + (-2)\bullet(-1)\right) = 1\,.$$

That is, when restricted from D_4 to $\mathbb{Z}_4$, the irreducible $L_{2\times 2}$ splits into two pieces: $\chi_{L_{2\times 2}} = \chi_{L_1} + \chi_{L_3}$, and $L_{2\times 2} = L_1 \oplus L_3$. This behavior is generic: these two component representations, which capture left- and right-rotation respectively, are intrinsically different within $\mathbb{Z}_4$, but there is a larger symmetry, included in D_4, which brings them together -- the mirror elements.