Exam, 2014-2015

Do a total of 30 points (more if you want, of course). Show your work!

Q1: 10 points (4 parts, 2 points each for the first two parts, 3 points each for the second two parts)
Q2: 10 points (11 parts, 1 point each, max credit 10 points)
Q3: 10 points (5 parts, 2 points each)
Q4: 12 points (6 parts, 2 points each)

Q1: Constructing a group representation based on cosets.

Let $G$ be a finite group, and $H$ a subgroup of $G$. Recall that the coset $Hb$ is the set of all of the elements $g$ of $G$ that can be written in the form $g = h \cdot b$, for some element $h$ in $H$, and that the cosets constitute a partition of $G$ into disjoint subsets.

Now consider the free vector space $W$ on the cosets: that is, $W$ be the vector space of functions from the cosets $Hb$ to $\mathbb{C}$. Since $W$ is a free vector space, it has a natural inner product, $\langle x, y \rangle = \sum_{Hb} x(Hb) \overline{y(Hb)}$

For any function $x$ in $W$ and any element $p$ of $G$, we define $Q_p$, a member of $\text{Hom}(W, W)$, as follows: $Q_p$ takes $x$ (a function on the cosets) to the $Q_p(x)$ (another function on the cosets) whose value at the coset $Hb$ is given by $\left(Q_p(x)\right)(Hb) = x(Hbp)$.

A. Show that this is a group representation.
B. What is its dimension (in terms of the sizes $\#(G)$ of $G$, and $\#(H)$ of $H$)?
C. Suppose further that $G$ is commutative. What is the character of $Q$?
D. (No longer supposing that $G$ is commutative). Under what circumstances is $Q$ irreducible?
Question 2. Permutation matrices and eigendecompositions

Consider an $n \times n$ cyclic permutation matrix $M$, defined by $m_{j,j+1} = 1$ for $1 \leq j \leq (n-1)$, $m_{n,1} = 1$, and otherwise zero. Here, for $n = 5$: 

$$
M = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

A. Is $M$ unitary?
B. Is $M$ self-adjoint?
C. What are the eigenvalues of $M$?
D. What are the eigenvectors of $M$?
E. Consider a matrix $C$, which is cyclic but with entries not necessarily drawn from $\{0,1\}$. (so it is not a permutation matrix). Specifically, $c_{j,k} = f_{k-j}$, where $k-j$ is interpreted mod $n$. For example, for $n = 5$,

$$
C = \begin{pmatrix}
f_0 & f_1 & f_2 & f_3 & f_4 \\
f_4 & f_0 & f_1 & f_2 & f_3 \\
f_2 & f_3 & f_4 & f_0 & f_1 \\
f_1 & f_2 & f_3 & f_4 & f_0
\end{pmatrix}. \text{ Is } C \text{ unitary?}
$$
F. Is $C$ self-adjoint?
G. Does $C$ commute with $M$?
H. What are the eigenvectors of $C$?
I. What are the eigenvalues of $C$?
J. What are the eigenvalues of this matrix, which is a permutation matrix but not cyclic?

$$
P = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

K. What can you say about the eigenvalues of any permutation matrix of size $n \times n$?
Question 3. Transfer function and power spectrum in a simple physical system

Say an output, $x(t)$, is related to an input, $s(t)$ by $m \frac{d^2 x}{dt^2} = s - kx - c \frac{dx}{dt}$. That is, the output $x$ is the position of an object of mass $m$ that is subject to a sum of three forces: the input signal $s$, a spring-like restoring force $-kx$, and a frictional force $-c \frac{dx}{dt}$.

A. Determine the transfer function that relates $x$ to $s$.

B. Say $s$ is white noise. What is the power spectrum of $x$?

C and D. Under what circumstances does the power spectrum have a peak at a nonzero frequency? At what frequency is the peak?

E. Describe what happens if $c = 0$ (no friction).
Question 4. Coherence and network identification

Say $S_1(t)$ and $S_2(t)$ are noise sources with power spectra $P_{S_1}(\omega)$ and $P_{S_2}(\omega)$, not necessarily independent, which are connected to two observable outputs $R_1(t)$ and $R_2(t)$ by the following network, where $L_{ij}$ are linear filters with transfer functions $\tilde{L}_ij(\omega)$.

A. Find the power spectra $P_{R_1}(\omega)$ and $P_{R_2}(\omega)$ of the outputs in terms of the power spectra $P_{S_1}(\omega)$ and cross-spectrum $P_{S_1S_2}(\omega)$ of the inputs.

B. Find the cross-spectrum of the outputs.

C and D. Same as A and B, but for $N$ independent noise sources fully connected to $N$ observable outputs, rather than two.

E. Define $S(\omega)$ as the “cross-spectral matrix” of $S$, i.e., the matrix whose elements are given by $S$

$$S_{ij}(\omega) = \begin{cases} P_{S_i}(\omega), & i = j \\ P_{S_iS_j}(\omega), & i \neq j \end{cases},$$

and similarly for $R$. Further define $\tilde{L}(\omega)$ as the matrix of transfer functions $\tilde{L}_ij(\omega)$.

Write a concise expression for $S(\omega)$ in terms of $R(\omega)$ and $\tilde{L}(\omega)$.

F. Now say we know that each of the noise inputs are independent, and have a flat power spectrum, specifically, that $S(\omega) = I$. Can we deduce the matrix $\tilde{L}(\omega)$ from the matrix $R(\omega)$? Why or why not?