Linear Transformations and Group Representations

Homework #1 (2016-2017), Answers

Q1: Another mapping from a group (the rotations of a circle) into linear operators. Here, $V$ is a two-dimensional vector space.

A. Find the eigenvalues of the transformation

$$R_\theta = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}.$$

Find the the characteristic equation:

$$\det(zI - R) = \det \begin{pmatrix}
z - \cos \theta & -\sin \theta \\
\sin \theta & z - \cos \theta
\end{pmatrix} = (z - \cos \theta)(z - \cos \theta) - (-\sin \theta)(\sin \theta)
$$

$$= z^2 - 2z \cos \theta + \cos^2 \theta + \sin^2 \theta$$

$$= z^2 - 2z \cos \theta + 1$$

The eigenvalues are the roots of the characteristic equation, which we find by the quadratic formula:

$$\lambda = \frac{2 \cos \theta \pm \sqrt{4 \cos^2 \theta - 4}}{2}$$

$$= \cos \theta \pm \sqrt{\cos^2 \theta - 1}$$

$$= \cos \theta \pm i \sin \theta$$

$$= e^{\pm i \theta}$$

So there are two eigenvalues, $e^{i \theta}$ and $e^{-i \theta}$.

B. Find its eigenvectors.

We seek vectors $\vec{x} = \begin{pmatrix} u \\ v \end{pmatrix}$ for which $R\vec{x} = e^{i \theta} \vec{x}$ (and also, $R\vec{x} = e^{-i \theta} \vec{x}$). Looking just at the eigenvalue $e^{i \theta}$,

$$R\vec{x} = e^{i \theta} \vec{x}$$

implies

$$\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = e^{i \theta} \begin{pmatrix} u \\ v \end{pmatrix},$$

i.e.,

$$u \cos \theta + v \sin \theta = e^{i \theta} u$$

and

$$-u \sin \theta + v \cos \theta = e^{i \theta} v.$$ Using $e^{i \theta} = \cos \theta + i \sin \theta$, the first equation is equivalent to $u \cos \theta + v \sin \theta = u \cos \theta + iv \sin \theta$, i.e., $v = iu$, and the second equation is equivalent to $-u \sin \theta + v \cos \theta = v \cos \theta + iv \sin \theta$, i.e., $-u = iv$. So both equations solve with $v = iu$, i.e., the eigenvector is $\vec{x}_+ = \begin{pmatrix} 1 \\ i \end{pmatrix}$ (and any multiple of it).

Similarly, for the eigenvalue $e^{-i \theta}$, the eigenvector is $\vec{x}_- = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ (and any multiple of it).
C. Since all of the transformations $R_\theta$ have the same eigenvectors (as shown in part B), they should commute. That is, $R_\theta R_\phi = R_\phi R_\theta$. Verify this.

$$R_\theta R_\phi = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos \theta \cos \phi - \sin \theta \sin \phi & \cos \theta \sin \phi + \sin \theta \cos \phi \\ -\sin \theta \cos \phi - \cos \theta \sin \phi & -\sin \theta \sin \phi + \cos \theta \cos \phi \end{pmatrix}.$$  

$$= \begin{pmatrix} \cos(\theta + \phi) & \sin(\theta + \phi) \\ -\sin(\theta + \phi) & \cos(\theta + \phi) \end{pmatrix} = R_{\theta+\phi}.$$

Therefore, $R_\theta R_\phi = R_{\theta+\phi} = R_{\phi+\theta} = R_\phi R_\theta$, so we have shown that $R_\theta R_\phi = R_\phi R_\theta$.

Q2: Eigenvalues and eigenvectors in a function space. Here, $V$ is the vector space of functions $f$ on the real line. Consider the mapping $H$, defined by $Hf(x) = \frac{d^2 f}{dx^2}(x) - x^2 f(x)$.

A. Show that $H$ is linear.

We need to show that $H$ preserves addition and scalar multiplication. To show that $H$ preserves addition:

$$H(f + g)(x) = \frac{d^2}{dx^2}((f + g)(x)) - x^2((f(x) + g(x))$$

$$= \frac{d^2 f}{dx^2}(x) + \frac{d^2 g}{dx^2}(x) - x^2 f(x) - x^2 g(x)$$

$$= \frac{d^2 f}{dx^2}(x) - x^2 f(x) + \frac{d^2 g}{dx^2}(x) - x^2 g(x)$$

$$= H(f)(x) + H(g)(x)$$

To show that $H$ preserves scalar multiplication:

$$H(\alpha f)(x) = \frac{d^2}{dx^2}(\alpha f(x)) - x^2\alpha f(x) = \alpha \left( \frac{d^2 f}{dx^2}(x) - x^2 f(x) \right) = \alpha H(f)(x).$$

B. Show that $u_0(x) = e^{-x^2/2}$ is an eigenvector of $H$, and find its eigenvalue.

If $u_0(x) = e^{-x^2/2}$, then $\frac{d}{dx} u_0(x) = -xe^{-x^2/2}$, and $\frac{d^2}{dx^2} u_0(x) = \frac{d}{dx} (-xe^{-x^2/2}) = x^2 e^{-x^2/2} - e^{-x^2/2}$. So

$$Hu_0(x) = \frac{d^2}{dx^2} u_0(x) - x^2 u_0(x) = -e^{-x^2/2} = -u_0(x),$$

so the eigenvalue is $-1$.

C. Show that $u_1(x) = xe^{-x^2/2}$ is an eigenvector of $H$, and find its eigenvalue.
If \( u_t(x) = xe^{-x^2/2} \), then \( \frac{d}{dx} u_t(x) = -x^2 e^{-x^2/2} + e^{-x^2/2} \), and

\[
\frac{d^2}{dx^2} u_t(x) = \frac{d}{dx} \left( -x^2 e^{-x^2/2} + e^{-x^2/2} \right) = x^3 e^{-x^2/2} - 2xe^{-x^2/2} - xe^{-x^2/2} = x^3 e^{-x^2/2} - 3xe^{-x^2/2}.
\]

So \( Hu_t(x) = \frac{d^2}{dx^2} u_t(x) - x^2 u_t(x) = -3xe^{-x^2/2} = -3u_t(x) \), so the eigenvalue is \(-3\).

**Q3. Eigenvalues of a permutation matrix.** Say \( M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \), so

\[
M = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.
\]

**A. Show that** \( M^3 = I \).

\[
M^3 = M^2 M = M^2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = M \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.
\]

(Or, we could just compute \( M^3 \) by matrix multiplication.)

**B. What are the eigenvalues of** \( M \)?

Say \( \nu \) is an eigenvector and \( \lambda \) is its eigenvalue. Then \( M^3 \nu = \nu \), since \( M^3 \) is the identity (by part A). But also, \( M^3 \nu = M^2 (M \nu) = M^2 \lambda \nu = \lambda M^2 \nu = (\lambda M)(M \nu) = (\lambda M)(\lambda \nu) = \lambda^3 M \nu = \lambda^3 \nu \). So \( \lambda^3 \nu = \nu \), and \( \lambda^3 = 1 \).

So the three eigenvalues are the roots of \( \lambda^3 = 1 \), namely, \( 1, e^{\frac{2\pi i}{3}}, \) and \( e^{\frac{4\pi i}{3}} \).