Groups, Fields, and Vector Spaces

Homework #3 (2018-2019), Answers

Q1: Tensor products: concrete examples – preliminary for determinant

Let V and W be two-dimensional vector spaces, with bases $\{v_1, v_2\}$ *and* $\{w_1, w_2\}$. *So* $\{v_i \otimes w_j\}$ *is a basis for* $V \otimes W$. *Say* $x_i \in V$ *has the basis expansion* $x = \alpha_1 v_1 + \alpha_2 v_2$ *and* $y_i \in W$ *has the basis expansion* $y = \beta_1 w_1 + \beta_2 w_2$.

A. Expand $x \otimes y$ *in the basis* $\{v_i \otimes w_j\}$.

$$
\begin{aligned}
& x \otimes y \\
& = (\alpha_1 v_1 + \alpha_2 v_2) \otimes (\beta_1 w_1 + \beta_2 w_2) \\
& = (\alpha_1 v_1 + \alpha_2 v_2) \otimes (\beta_1 w_1) + (\alpha_1 v_1 + \alpha_2 v_2) \otimes (\beta_2 w_2) \\
& = (\alpha_1 v_1) \otimes (\beta_1 w_1) + (\alpha_2 v_2) \otimes (\beta_1 w_1) + (\alpha_1 v_1) \otimes (\beta_2 w_2) + (\alpha_2 v_2) \otimes (\beta_2 w_2) \\
& = \alpha_1 \beta_1 (v_1 \otimes w_1) + \alpha_2 \beta_1 (v_2 \otimes w_1) + \alpha_1 \beta_2 (v_1 \otimes w_2) + \alpha_2 \beta_2 (v_2 \otimes w_2)
\end{aligned}
$$

B. Now say $V = W$, *and we are using the same basis for x and y, so that* $x = \alpha_1 v_1 + \alpha_2 v_2$ *and* $y = \beta_1 v_1 + \beta_2 v_2$. *Expand* $x \otimes y$ *in the basis* $\{v_i \otimes v_j\}$.

Taking $w_i = v_i$ in part A,

$$
x \otimes y = \alpha_1 \beta_1 (v_1 \otimes v_1) + \alpha_2 \beta_1 (v_2 \otimes v_1) + \alpha_1 \beta_2 (v_1 \otimes v_2) + \alpha_2 \beta_2 (v_2 \otimes v_2)
$$

C. Expand $x \otimes y + y \otimes x$ *in the basis* $\{v_i \otimes v_j\}$.

$$
\begin{aligned}
& (x \otimes y) + (y \otimes x) \\
& = \left(\alpha_1 \beta_1 (v_1 \otimes v_1) + \alpha_2 \beta_1 (v_2 \otimes v_1) + \alpha_1 \beta_2 (v_1 \otimes v_2) + \alpha_2 \beta_2 (v_2 \otimes v_2)\right) \\
& + \left(\beta_1 \alpha_1 (v_1 \otimes v_1) + \beta_2 \alpha_1 (v_2 \otimes v_1) + \beta_1 \alpha_2 (v_1 \otimes v_2) + \beta_2 \alpha_2 (v_2 \otimes v_2)\right) \\
& = 2\alpha_1 \beta_1 (v_1 \otimes v_1) + (\alpha_2 \beta_1 + \beta_2 \alpha_1)(v_2 \otimes v_1) + (\alpha_1 \beta_2 + \beta_1 \alpha_2)(v_1 \otimes v_2) + 2\alpha_2 \beta_2 (v_2 \otimes v_2)
\end{aligned}
$$

D. Expand $x \otimes y - y \otimes x$ *in the basis* $\{v_i \otimes v_j\}$.

$$
\begin{aligned}
& (x \otimes y) - (y \otimes x) \\
& = \left(\alpha_1 \beta_1 (v_1 \otimes v_1) + \alpha_2 \beta_1 (v_2 \otimes v_1) + \alpha_1 \beta_2 (v_1 \otimes v_2) + \alpha_2 \beta_2 (v_2 \otimes v_2)\right) \\
& - \left(\beta_1 \alpha_1 (v_1 \otimes v_1) + \beta_2 \alpha_1 (v_2 \otimes v_1) + \beta_1 \alpha_2 (v_1 \otimes v_2) + \beta_2 \alpha_2 (v_2 \otimes v_2)\right) \\
& = (\alpha_2 \beta_1 - \beta_2 \alpha_1)(v_2 \otimes v_1) + (\alpha_1 \beta_2 - \beta_1 \alpha_2)(v_1 \otimes v_2) \\
& = (\alpha_2 \beta_1 - \beta_2 \alpha_1)\left((v_2 \otimes v_1) - (v_1 \otimes v_2)\right)
\end{aligned}
$$

Q2. Explicit construction of a 2 x 2 determinant

A similar setup of part B of Q1: $\{v_1, v_2\}$ *is a basis for a two-dimensional space V. In this basis, the linear transformation* M *is defined by the matrix* $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$, *where* $Mv_1 = m_{11}v_1 + m_{12}v_2$ *and* $Mv_2 = m_{21}v_1 + m_{22}v_2$. *We are guaranteed that* $anti(V^{\otimes 2})$ is one-dimensional, and that it is spanned by $\varphi = (v_1 \otimes v_2) - (v_2 \otimes v_1)$, so that $M\varphi$ is a multiple of φ. Compute this multiple, i.e., the determinant, by computing $M\varphi$.

First, using the definition of M *and how it acts in a tensor-product space:*

$$M\varphi = M(v_1 \otimes v_2) - M(v_2 \otimes v_1)$$
$$= (Mv_1 \otimes Mv_2) - (Mv_2 \otimes Mv_1)$$
$$= ((m_{11}v_1 + m_{12}v_2) \otimes (m_{21}v_1 + m_{22}v_2)) - ((m_{21}v_1 + m_{22}v_2) \otimes (m_{11}v_1 + m_{12}v_2))$$

Now, reduce the above using the properties of tensor product (i.e., the distributive law):

$$((m_{11}v_1 + m_{12}v_2) \otimes (m_{21}v_1 + m_{22}v_2)) - ((m_{21}v_1 + m_{22}v_2) \otimes (m_{11}v_1 + m_{12}v_2))$$
$$= (m_{11}m_{21}(v_1 \otimes v_1) + m_{11}m_{22}(v_1 \otimes v_2) + m_{12}m_{21}(v_2 \otimes v_1) + m_{12}m_{22}(v_2 \otimes v_2))$$
$$-(m_{21}m_{11}(v_1 \otimes v_1) + m_{21}m_{12}(v_1 \otimes v_2) + m_{22}m_{11}(v_2 \otimes v_1) + m_{22}m_{12}(v_2 \otimes v_2))$$
$$= (m_{11}m_{21} - m_{21}m_{11})(v_1 \otimes v_1) + (m_{11}m_{22} - m_{21}m_{12})(v_1 \otimes v_2)$$
$$+(m_{12}m_{21} - m_{22}m_{11})(v_2 \otimes v_1) + (m_{12}m_{22} - m_{22}m_{12})(v_2 \otimes v_2)$$
$$= (m_{11}m_{22} - m_{21}m_{12})(v_1 \otimes v_2) + (m_{12}m_{21} - m_{22}m_{11})(v_2 \otimes v_1)$$
$$= (m_{11}m_{22} - m_{21}m_{12})((v_1 \otimes v_2) - (v_2 \otimes v_1))$$
$$= (m_{11}m_{22} - m_{21}m_{12})\varphi$$

This last factor, as expected, is the determinant of M.

Q3. Another finite field example

Recall that $\mathbb{Z}_2$ *is the field containing* $\{0,1\}$, *with addition and multiplication defined (mod 2). Consider the polynomial* $x^4 + x + 1 = 0$. *This has no solutions in* $\mathbb{Z}_2$, *so let's add a formal quantity* ξ *for which* $\xi^4 + \xi + 1 = 0$ *(and which satisfies the associative, commutative, and distributive laws for addition and multiplication with itself and with* $\{0,1\}$*), and see whether it generates a field.*

Using $\xi^4 + \xi + 1 = 0$, *express* ξ^r *in terms of 1,* ξ, ξ^2, *and* ξ^3 *for* $r = 1,...,15$.

Since field operations are "mod 2", we can replace -1 by +1, and 0 by 2. So, for example, $\xi^4 + \xi + 1 = 0$ implies $\xi^4 = \xi + 1$. Using the field properties (distributive law),
$\xi^5 = \xi \cdot \xi^4 = \xi(\xi + 1) = \xi^2 + \xi$;

$\xi^6 = \xi \cdot \xi^5 = \xi(\xi^2 + \xi) = \xi^3 + \xi^2$;
$\xi^7 = \xi \cdot \xi^6 = \xi(\xi^3 + \xi^2) = \xi^4 + \xi^3 = \xi^3 + \xi + 1$ (Here, we had to use $\xi^4 = \xi + 1$ in the last step.)

Working similarly, the table of coefficients is:

	ξ^3	ξ^2	ξ^1	ξ^0
$\xi^0 =$	0	0	0	1
$\xi^1 =$	0	0	1	0
$\xi^2 =$	0	1	0	0
$\xi^3 =$	1	0	0	0
$\xi^4 =$	0	0	1	1
$\xi^5 =$	0	1	1	0
$\xi^6 =$	1	1	0	0
$\xi^7 =$	1	0	1	1
$\xi^8 =$	0	1	0	1
$\xi^9 =$	1	0	1	0
$\xi^{10} =$	0	1	1	1
$\xi^{11} =$	1	1	1	0
$\xi^{12} =$	1	1	1	1
$\xi^{13} =$	1	1	0	1
$\xi^{14} =$	1	0	0	1
$\xi^{15} =$	0	0	0	1

Note that every combination of 0's and 1's occurs in some row, except for 0,0,0,0. (Why does this have to be?) Note also that $\xi^{15} = \xi^0 = 1$.