Q1: Impulse responses and transfer functions

A. Exponential decay: For a system $F$ with impulse response $f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$, find the transfer function $\hat{f}(\omega)$.

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt = \int_{0}^{\infty} e^{-i\omega t} \lambda e^{-\lambda t} dt = \lambda \int_{0}^{\infty} e^{-(i\omega + \lambda)t} dt$$

$$= -\left( \frac{\lambda}{i\omega + \lambda} \right) e^{-(i\omega + \lambda)t} \bigg|_{0}^{\infty} = \frac{\lambda}{i\omega + \lambda} = \frac{1}{1 + i\omega / \lambda}.$$

B. Boxcar average: For a system $B_h$ with impulse response $b_h(t) = \begin{cases} \frac{1}{h}, & t \in [0,h] \\ 0, & \text{otherwise} \end{cases}$, find the transfer function $\hat{b}_h(\omega)$.

$$\hat{b}_h(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} b_h(t) dt = \frac{1}{h} \int_{0}^{h} e^{-i\omega t} dt = \frac{1}{i\omega h} e^{-i\omega h} \bigg|_{0}^{h} = \frac{1 - e^{-i\omega h}}{i\omega h}.$$

It can be useful to rewrite this as follows, to separate the phase and amplitude:

$$\hat{b}_h(\omega) = \frac{1 - e^{-i\omega h}}{i\omega h} = \frac{e^{-i\omega h/2}}{i\omega h} \left( e^{i\omega h/2} - e^{-i\omega h/2} \right) = \frac{e^{-i\omega h/2}}{\omega h / 2} \left( e^{i\omega h/2} - e^{-i\omega h/2} \right)$$

$$= e^{-i\omega h/2} \frac{\sin(\omega h / 2)}{\omega h / 2} = e^{-i\omega h/2} \text{sinc}(\omega h / 2),$$

where the “sinc” function is defined by $\text{sinc}(u) = \frac{\sin(u)}{u}$. Note that $\text{sinc}(0) = 1$ and the sinc function has zeros at $u = \pm n\pi$, for nonzero $n$. This means that the “boxcar” average is insensitive to input components that repeat an integer number of times in the averaging interval $h$.

C. Pure delay: For a system $F_T$ with impulse response $f_T(t) = \delta(t - T)$, find the transfer function $\hat{f}_T(\omega)$.

$$\hat{f}_T(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f_T(t) dt = \int_{0}^{\infty} e^{-i\omega t} \delta(t - T) dt = e^{-i\omega T}.$$

D. Differentiation: Consider a system $F_{\text{diff}}$ whose output is the derivative of the input. We can’t write an impulse response for this system in a straightforward way, because the derivative of a delta-function is not defined. But we can determine its transfer function, by considering its response to sinusoids $e^{i\omega t}$. What is its transfer function $\hat{f}_{\text{diff}}(\omega)$?

$F_{\text{diff}}$ takes $e^{i\omega t}$ into $\frac{d}{dt} e^{i\omega t} = i\omega e^{i\omega t}$. So $F_{\text{diff}}$ multiplies $e^{i\omega t}$ by $i\omega$. Therefore, $\hat{f}_{\text{diff}}(\omega) = i\omega$. 
Note that \( F_{\text{diff}} = \lim_{T \to 0} \frac{f_0 - f_T}{T} \) (the definition of the derivative). Correspondingly,

\[
\lim_{T \to 0} \frac{\hat{f}_0(\omega) - \hat{f}_T(\omega)}{T} = \lim_{T \to 0} \frac{1 - e^{-i\omega T}}{T} = \lim_{T \to 0} \frac{1 - (1 - i\omega T)}{T} = \lim_{T \to 0} \frac{i\omega T}{T} = i\omega = \hat{f}_{\text{diff}}(\omega)
\]

Q2: Biased diffusion

In the notes, we modeled diffusion as a random walk from \( x = 0 \) to \( x = \pm b \), with equal probability, in time \( \Delta T \). That is, \( F_{\Delta T}(x) = \frac{1}{2}(\delta(x-b) + \delta(x+b)) \). We saw that this had a stable limit as \( \Delta T \to 0 \) if \( b^2 = A\Delta T \), i.e., \( b = \sqrt{A\Delta T} \).

Now consider a biased process, in which the probability of a step to \( +b \) is \( \frac{1}{2}(1 + \alpha) \) and the probability of a step to \( -b \) is \( \frac{1}{2}(1 - \alpha) \). So now, \( F_{\Delta T}(x) = \frac{1}{2}((1 + \alpha)\delta(x-b) + (1 - \alpha)\delta(x+b)) \).

A. Determine \( \hat{F}_{\Delta T}(\omega) \).

\[
\hat{F}_{\Delta T}(\omega) = \int_{-\infty}^{\infty} F_{\Delta T}(x)e^{-i\omega x}dx = \frac{1}{2} \int_{-\infty}^{\infty} ((1 + \alpha)\delta(x-b) + (1 + \alpha)\delta(x+b))e^{-i\omega x}dx
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} (\delta(x-b) + \delta(x+b))e^{-i\omega x}dx + \frac{\alpha}{2} \int_{-\infty}^{\infty} (\delta(x-b) - \delta(x+b))e^{-i\omega x}dx
\]

\[
= \frac{1}{2}(e^{-i\omega b} + e^{i\omega b}) + \frac{\alpha}{2}(e^{-i\omega b} - e^{i\omega b}) = \cos(\omega b) - i\alpha \sin(\omega b)
\]

B. How should \( \alpha \) vary with \( \Delta T \) to ensure a stable limit for \( \hat{F}_T(\omega) = \lim_{\Delta T \to 0} \left( \hat{F}_{\Delta T}(\omega) \right)^{T/\Delta T} \) as \( \Delta T \to 0 \), and what is this limit?

For small \( b \)

\[
\hat{F}_{\Delta T}(\omega) = \cos(\omega b) - i\alpha \sin(\omega b) \approx 1 - \frac{1}{2}b^2\omega^2 - i\alpha\omega b = e^{-\frac{1}{2}b^2\omega^2 - i\omega b}.
\]

So

\[
\left( \hat{F}_{\Delta T}(\omega) \right)^{T/\Delta T} \approx e^{\frac{1}{2}b^2\omega^2 - i\omega b} \left( \frac{T}{\Delta T} \right).
\]

We need \( b^2 / \Delta T \) and \( \alpha b / \Delta T \) to have a stable limit as \( \Delta T \to 0 \). So \( b^2 / \Delta T = A \) implies \( b = \sqrt{A\Delta T} \), and \( \alpha b / \Delta T = h \) implies \( \alpha = h\Delta T / b = \frac{h\Delta T}{\sqrt{A\Delta T}} = h\sqrt{\frac{\Delta T}{A}} \).

So

\[
\hat{F}_T(\omega) = \lim_{\Delta T \to 0} \left( \hat{F}_{\Delta T}(\omega) \right)^{T/\Delta T} = e^{\frac{1}{2}b^2\omega^2 - i\omega b} \left( \frac{T}{\Delta T} \right) = e^{-\omega^2 AT/2 - i\omega b T} = e^{-\omega^2 AT/2 - i\omega h T}
\]

C. If, at time 0, the distribution is \( p_0(x) = \delta(x) \), what is the distribution \( p_T(x) \) at time \( T \)?

\[
\hat{p}_T(\omega) = \hat{F}_T(\omega)\hat{p}(0), \text{ and since } p_0(x) = \delta(x), \hat{p}(0) = 1. \text{ So } \hat{p}_T(\omega) = \hat{F}_T(\omega) = e^{-\omega^2 AT/2 + i\omega h T}. \text{ Inverting the transform,}
\]
\[ p_T(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}_T(\omega) e^{j\omega x} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2 AT^2 / 2 + i\omega x} e^{j\omega x} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2 AT^2 / 2} e^{j\omega(x-hT)} d\omega. \]

The final integral is the same as the one that arose for unbiased diffusion (notes), but with \( x - hT \) replacing \( x \).

So \[ p_T(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}_T(\omega) e^{j\omega x} d\omega = \frac{1}{\sqrt{2\pi AT}} e^{-(x-hT)^2 / 2AT}, \] a Gaussian centered at \( x = hT \) whose variance is \( \sqrt{AT} \).

Not surprisingly, when the probabilities of leftward and rightward steps are unequal, the distribution drifts by an amount proportional to time. Drift and broadening of the distribution do not interact.