Linear Systems, Black Boxes, and Beyond

Homework #1 (2018-2019), Answers

Q1: Impulse responses and transfer functions

A. Exponential decay: For a system F *with impulse response* $f(t)=\begin{cases}\lambda e^{-\lambda t}, & t \ge 0\\ 0, & t<0\end{cases}$*, find the transfer function* $\hat{f}(\omega)$.

$$\hat{f}(\omega)=\int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt = \int_{0}^{\infty} e^{-i\omega t}\lambda e^{-\lambda t}dt = \lambda\int_{0}^{\infty} e^{-(i\omega+\lambda)t}dt$$
$$= -\left(\frac{\lambda}{i\omega+\lambda}\right)e^{-(i\omega+\lambda)t}\Big|_0^{\infty} = \frac{\lambda}{i\omega+\lambda} = \frac{1}{1+i\omega/\lambda}.$$

B. Boxcar average: For a system B_h *with impulse response* $b_h(t)=\begin{cases}\frac{1}{h}, & t\, in[0,h]\\ 0, & otherwise\end{cases}$*, find the transfer function* $\hat{b}_h(\omega)$.

$$\hat{b}_h(\omega)=\int_{-\infty}^{\infty} e^{-i\omega t} b_h(t)dt = \frac{1}{h}\int_{0}^{h} e^{-i\omega t}dt = -\frac{1}{i\omega h}e^{-i\omega t}\Big|_0^{h} = \frac{1-e^{-i\omega h}}{i\omega h}.$$

It can be useful to rewrite this as follows, to separate the phase and amplitude:

$$\hat{b}_h(\omega)=\frac{1-e^{-i\omega h}}{i\omega h} = \frac{e^{-i\omega h/2}}{i\omega h}\left(e^{i\omega h/2}-e^{-i\omega h/2}\right) = \frac{e^{-i\omega h/2}}{\omega h/2}\left(\frac{e^{i\omega h/2}-e^{-i\omega h/2}}{2i}\right)$$
$$= e^{-i\omega h/2}\frac{\sin(\omega h/2)}{\omega h/2} = e^{-i\omega h/2}\operatorname{sinc}(\omega h/2)$$

, where the "sinc" function is defined by $\operatorname{sinc}(u)=\frac{\sin u}{u}$. Note that $\operatorname{sinc}(0)=1$ and the sinc function has zeros at $u=\pm n\pi$, for nonzero n. This means that the "boxcar" average is insensitive to input components that repeats an integer number of times in the averaging interval h.

C. Pure delay: For a system F_T *with impulse response* $f_T(t)=\delta(t-T)$*, find the transfer function* $\hat{f}_T(\omega)$.

$$\hat{f}_T(\omega)=\int_{-\infty}^{\infty} e^{-i\omega t} f_T(t)dt = \int_{0}^{\infty} e^{-i\omega t}\delta(t-T)dt = e^{-i\omega T}$$

D. Differentiation: Consider a system F_{diff} *whose output is the derivative of the input. We can't write an impulse response for this system in a straightforward way, because the derivative of a delta-function is not defined. But we can determine its transfer function, by considering its response to sinusoids* $e^{i\omega t}$*. What is its transfer function* $\hat{f}_{diff}(\omega)$*?*

F_{diff} takes $e^{i\omega t}$ into $\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t}$. So F_{diff} multiplies $e^{i\omega t}$ by $i\omega$. Therefore, $\hat{f}_{diff}(\omega)=i\omega$.

Note that $F_{diff} = \lim_{T\to 0} \frac{f_0 - f_T}{T}$ (the definition of the derivative). Correspondingly,

$$\lim_{T\to 0} \frac{\hat{f}_0(\omega)-\hat{f}_T(\omega)}{T} = \lim_{T\to 0} \frac{1-e^{-i\omega T}}{T} = \lim_{T\to 0} \frac{1-(1-i\omega T)}{T} = \lim_{T\to 0} \frac{i\omega T}{T} = i\omega = \hat{f}_{diff}(\omega)$$

Q2: Biased diffusion

In the notes, we modeled diffusion as a random walk from $x=0$ *to* $x=\pm b$ *, with equal probability, in time* ΔT *. That is,* $F_{\Delta T}(x) = \frac{1}{2}\left(\delta(x-b)+\delta(x+b)\right)$*. We saw that this had a stable limit as* $\Delta T \to 0$ *if* $b^2 = A\Delta T$ *, i.e.,* $b = \sqrt{A\Delta T}$ *.*

Now consider a biased process, in which the probability of a step to $+b$ *is* $\frac{1}{2}(1+\alpha)$ *and the probability of a step to* $-b$ *is* $\frac{1}{2}(1-\alpha)$ *.So now,* $F_{\Delta T}(x) = \frac{1}{2}\left((1+\alpha)\delta(x-b)+(1-\alpha)\delta(x+b)\right)$*.*
A. Determine $\hat{F}_{\Delta T}(\omega)$*.*

$$\hat{F}_{\Delta T}(\omega) = \int_{-\infty}^{\infty} F_{\Delta T}(x) e^{-i\omega x} dx = \frac{1}{2}\int_{-\infty}^{\infty} \left((1+\alpha)\delta(x-b)+(1+\alpha)\delta(x+b)\right) e^{-i\omega x} dx$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} \left(\delta(x-b)+\delta(x+b)\right) e^{-i\omega x} dx + \frac{\alpha}{2}\int_{-\infty}^{\infty} \left(\delta(x-b)-\delta(x+b)\right) e^{-i\omega x} dx$$
$$= \frac{1}{2}\left(e^{-i\omega b}+e^{i\omega b}\right) + \frac{\alpha}{2}\left(e^{-i\omega b}-e^{i\omega b}\right) = \cos(\omega b) - i\alpha \sin(\omega b)$$

B. How should α *vary with* ΔT *to ensure a stable limit for* $\hat{F}_T(\omega) = \lim_{\Delta T\to 0}\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T}$ *as* $\Delta T \to 0$*, and what is this limit?*

For small b
$$\hat{F}_{\Delta T}(\omega) = \cos(\omega b) - i\alpha\sin(\omega b) \approx 1 - \frac{1}{2}b^2\omega^2 - i\alpha\omega b \approx e^{-\frac{1}{2}b^2\omega^2 - i\alpha\omega b}.$$
So
$$\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T} \approx e^{\left(-\frac{1}{2}b^2\omega^2 - i\alpha\omega b\right)\frac{T}{\Delta T}}.$$
We need $b^2/\Delta T$ and $\alpha b/\Delta T$ to have a stable limit as $\Delta T \to 0$. So $b^2/\Delta T = A$ implies $b = \sqrt{A\Delta T}$, and $\alpha b/\Delta T = h$ implies $\alpha = h\Delta T/b = \frac{h\Delta T}{\sqrt{A\Delta T}} = h\sqrt{\frac{\Delta T}{A}}$.

So $\hat{F}_T(\omega) = \lim_{\Delta T\to 0}\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T} = e^{\left(-\frac{1}{2}b^2\omega^2 - i\alpha\omega b\right)\frac{T}{\Delta T}} = e^{-\omega^2 AT/2 - i\omega\alpha bT/\Delta T} = e^{-\omega^2 AT/2 - i\omega hT}$

C. If, at time 0, the distribution is $p_0(x) = \delta(x)$*, what is the distribution* $p_T(x)$ *at time* T *?*
$\hat{p}_T(\omega) = \hat{F}_T(\omega)\hat{p}(0)$, and since $p_0(x) = \delta(x)$, $\hat{p}(0) = 1$. So $\hat{p}_T(\omega) = \hat{F}_T(\omega) = e^{-\omega^2 AT/2 + i\omega hT}$. Inverting the transform,

$$p_T(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{p}_T(\omega) e^{i\omega x} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\omega^2 AT/2 - i\omega hT} e^{i\omega x} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\omega^2 AT/2} e^{i\omega(x-hT)} d\omega .$$

The final integral is the same as the one that arose for unbiased diffusion (notes), but with $x - hT$ replacing x.

So $p_T(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{p}_T(\omega) e^{i\omega x} d\omega = \frac{1}{\sqrt{2\pi AT}} e^{-(x-hT)^2/2AT}$, a Gaussian centered at $x = hT$ whose variance is $\sqrt{AT}$.

Not surprisingly, when the probabilities of leftward and rightward steps are unequal, the distribution drifts by an amount proportional to time. Drift and broadening of the distribution do not interact.