Linear Systems, Black Boxes, and Beyond

Homework #1 (2018-2019), Questions

Q1: Impulse responses and transfer functions

A. Exponential decay: For a system F with impulse response $f(t)=\begin{cases}\lambda e^{-\lambda t}, & t\ge 0\\ 0, & t<0\end{cases}$, find the transfer function $\hat{f}(\omega)$.

B. Boxcar average: For a system B_h with impulse response $b_h(t)=\begin{cases}\frac{1}{h}, & t\, in[0,h]\\ 0, & otherwise\end{cases}$, find the transfer function $\hat{b}_h(\omega)$.

C. Pure delay: For a system F_T with impulse response $f_T(t)=\delta(t-T)$, find the transfer function $\hat{f}_T(\omega)$.

D. Differentiation: Consider a system F_{diff} whose output is the derivative of the input. We can't write an impulse response for this system in a straightforward way, because the derivative of a delta-function is not defined. But we can determine its transfer function, by considering its response to sinusoids $e^{i\omega t}$. What is its transfer function $\hat{f}_{diff}(\omega)$?

Q2: Biased diffusion

In the notes, we modeled diffusion as a random walk from $x=0$ to $x=\pm b$, with equal probability, in time ΔT. That is, $F_{\Delta T}(x)=\frac{1}{2}\left(\delta(x-b)+\delta(x+b)\right)$. We saw that this had a stable limit as $\Delta T\to 0$ if $b^2=A\Delta T$, i.e., $b=\sqrt{A\Delta T}$.

Now consider a biased process, in which the probability of a step to $+b$ is $\frac{1}{2}(1+\alpha)$ and the probability of a step to $-b$ is $\frac{1}{2}(1-\alpha)$ .So now, $F_{\Delta T}(x)=\frac{1}{2}\left((1+\alpha)\delta(x-b)+(1-\alpha)\delta(x+b)\right)$.

A. Determine $\hat{F}_{\Delta T}(\omega)$.

B. How should α vary with ΔT to ensure a stable limit for $\hat{F}_T(\omega)=\lim_{\Delta T\to 0}\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T}$ as $\Delta T\to 0$, and what is this limit?

C. If, at time 0, the distribution is $p_0(x)=\delta(x)$, what is the distribution $p_T(x)$ at time T?