Linear Transformations and Group Representations

Homework #2 (2018-2019), Answers

Q1: Some representations of the "continuous dihedral" group.

Let G be the "continuous dihedral" group , i.e., the group of rotations and reflections of a circle. For definiteness, let R_θ be a clockwise rotation by θ, and let M be the reflection in the vertical axis (that sends x to $-x$ and preserves y). The group consists of R_θ, M, and all the transformations that can be generated by composing them.

A. Verify geometrically that these group elements satisfy $R_\theta R_\phi = R_{\theta+\phi}$, $R_\theta M = MR_{-\theta}$, and $M^2 = I$ (the identity).

$R_\theta R_\phi = R_{\theta+\phi}$: $R_\theta R_\phi$ is a rotation by an angle ϕ followed by a rotation by an angle θ; the net result is rotation by the total angle $\theta + \phi$.

$R_\theta M = MR_{-\theta}$: Consider a point on the circle at an angle α from the vertical.
Result of applying $R_\theta M$: M moves the point to $-\alpha$; then R_θ moves the point to $-\alpha+\theta$.
Result of applying $MR_{-\theta}$: $R_{-\theta}$ moves the point to $\alpha-\theta$; then M moves the point to $-(\alpha-\theta) = -\alpha+\theta$, which is the same as the result for applying $R_\theta M$.

$M^2 = I$ *: Since* M is a mirroring, this is immediate.

B. Show that any element of the group is equal either to R_ϕ or $R_\phi M$, for some ϕ.
Since $M^2 = I$, any group element can be written $R_{\theta_1} M R_{\theta_2} M ... R_{\theta_k}$ or a similar product preceded or followed by M . Using $MR_\theta = R_{-\theta} M$, we can move every R from the right side of an M to its left, and then condense the resulting pair of M 's, e.g., $R_{\theta_1} M R_{\theta_2} M = R_{\theta_1} R_{-\theta_2} MM = R_{\theta_1-\theta_2}$, resulting in a reduction in the number of M 's. Repeating these swaps leads to a group element that is either of the form R_ϕ or $R_\phi M$,depending on whether the original number of M 's was even or odd.

*C. Geometrically, what is the transformation $R_\theta M R_\theta^{-1}$?What is its reduction to the form specified in part **B**?*

This is a mirror about a line that is rotated by θ from the vertical. Two ways. (i) One can think of $R_\theta M R_\theta^{-1}$ as an automorphism of the group, corresponding to a relabeling by rotation. (ii) One can follow a generic point that starts an angle α from the vertical. This point is transformed by R_θ^{-1} to $\alpha-\theta$, then by M to $-(\alpha-\theta) = -\alpha+\theta$, and then by R_θ to $(-\alpha+\theta)+\theta = -\alpha+2\theta$. So the starting angle α and the final angle $-\alpha+2\theta$ are equal distances from θ, as required for a mirror about a lie at an orientation θ .

Reduction via Part B: $R_\theta M R_\theta^{-1} = R_\theta M R_{-\theta} = R_\theta R_\theta M = R_{2\theta} M$.

D. Write R_θ *and* M *as* 2×2 *matrices, and thereby construct a 2-dimensional unitary representation* L *of* G *. Verify the identities of part A algebraically.*

R_θ : R_θ takes $\begin{pmatrix}1\\0\end{pmatrix}$, on the horizontal axis, to $\begin{pmatrix}\cos\theta\\-\sin\theta\end{pmatrix}$, and $\begin{pmatrix}0\\1\end{pmatrix}$, on the vertical axis, to $\begin{pmatrix}\sin\theta\\\cos\theta\end{pmatrix}$. So

$$R_\theta = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}.$$

$M : M$ takes $\begin{pmatrix}1\\0\end{pmatrix}$, on the horizontal axis, to $\begin{pmatrix}-1\\0\end{pmatrix}$, and $\begin{pmatrix}0\\1\end{pmatrix}$, on the vertical axis, to itself. So

$$M = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

$R_\theta R_\phi = R_{\theta+\phi}$:

$$R_\theta R_\phi = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix} = \begin{pmatrix}\cos\theta\cos\phi - \sin\theta\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi\\ -\sin\theta\cos\phi - \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi\end{pmatrix}$$

$$= \begin{pmatrix}\cos(\theta+\phi) & \sin(\theta+\phi)\\ -\sin(\theta+\phi) & -\cos(\theta+\phi)\end{pmatrix} = R_{\theta+\phi}$$

$R_\theta M = M R_{-\theta}$:

$$R_\theta M = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}-\cos\theta & \sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

$$M R_{-\theta} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} = \begin{pmatrix}-\cos\theta & \sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

$M^2 = I$:

$$M^2 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

E. What is the character of R_θ , $R_\theta M$ *, and* $R_\theta M R_\theta^{-1}$ *in the representation L?*

$$\chi_L(R_\theta) = tr(L_{R_\theta}) = tr\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} = 2\cos\theta .$$

$$\chi_L(M) = tr(L_M) = tr\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = 0 .$$

$$\chi_L(R_\theta M R_\theta^{-1}) = tr(L_{R_\theta M R_\theta^{-1}}) = tr(L_{R_\theta} L_M L_{R_\theta^{-1}}) = tr(L_{R_\theta} L_M (L_{R_\theta})^{-1}) = tr(L_M) = 0 .$$

F. Define $L^{[n]}_{R_\theta} = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix} = R^n = R_{n\theta}$ *and* $L^{[n]}_M = M$ *(the latter is independent of* n *). Show that* $L^{[n]}$ *is a representation. Note that to do this, it suffices to show that the mapping from group elements to the unitary matrices defined by* $L^{[n]}$ *will preserve the rules that govern group operations:* $R_\theta R_\phi = R_{\theta+\phi}$, $R_\theta M = MR_{-\theta}$, *and* $M^2 = I$.

$L^{[n]}_{R_\theta} L^{[n]}_{R_\phi} = R_{n\theta} R_{n\phi} = R_{n(\theta+\phi)} = L^{[n]}_{R_\theta R_\phi}$
$L^{[n]}_{R_\theta} L^{[n]}_M = R_{n\theta} M = MR_{-n\theta} = L^{[n]}_M L^{[n]}_{R_{-\theta}}$.
$\left(L^{[n]}_M\right)^2 = M^2 = I = L^{[n]}_I$.

G. Define $S_{R_\theta} = 1$ *and* $S_M = -1$. *Show S is a one-dimensional representation.*
Use the strategy in Part F.
$S_{R_\theta} S_{R_\phi} = 1 \bullet 1 = 1$, independent of θ or ϕ.
$S_{R_\theta} S_M = 1 \bullet (-1) = -1$, and $S_M S_{R_{-\theta}} = (-1) \bullet 1 = -1$
$\left(S_M\right)^2 = (-1)^2 = 1 = S_I$.

Q2: Characters of representations of a permutation group.

Let P be the permutation group on three objects. This has six elements.

A. Write each group element as a 3×3 *permutation matrix. As discussed, this is a unitary representation, which we can call* U *. For each of the six permutations* σ *, determine the character* $\chi_U(\sigma)$.

The character, which is the trace, counts the number of 1's on the diagonal, which is also the number of objects that are not affected by the permutation.

Group element σ	identity	(AB)	(AC)	(BC)	(ABC)	(ACB)
Permutation matrix U_σ	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
Character $\chi_U(\sigma) = tr(U_\sigma)$	3	1	1	1	0	0

B. Consider the subgroup of G *generated by* R_θ *and* M *, where* θ *is restricted to* 0 *,* $2\pi/3$ *, and* $4\pi/3$ *. Show that this is the permutation group on 3 objects.*

Each of these group operations map the 3 points of an equilateral triangle with vertices at angles 0, $2\pi/3$, and $4\pi/3$ to itself (say, A, B, and C). $R_{\frac{2\pi}{3}}$ corresponds to the permutation (ABC) and M corresponds to the permutation (BC). Products of M and $R_{\frac{2\pi}{3}}$ generate the remaining permutations.

C. Restricting the group representation L of Question 1 (parts D and E) to the subgroup in part B yields a 2-dimensional unitary representation of P. Determine its character for the six group elements of P.

We read off these values from Question 1 part E:

Group element σ	identity	(AB)	(AC)	(BC)	(ABC)	(ACB)
Equivalent in G	I	$R_{\frac{4\pi}{3}} M R_{\frac{4\pi}{3}}^{-1}$	$R_{\frac{2\pi}{3}} M R_{\frac{2\pi}{3}}^{-1}$	M	$R_{\frac{2\pi}{3}}$	$R_{\frac{4\pi}{3}}$
Character $\chi_L(\sigma) = tr(L_\sigma)$	2	0	0	0	-1	-1

D. Restricting the group representation S of Question 1 (part G) to the subgroup in part B yields a 1-dimensional unitary representation of P. Determine its character for the six group elements of P.

We read off these values from Question 1 part G:

Group element σ	identity	(AB)	(AC)	(BC)	(ABC)	(ACB)
Equivalent in G	I	$R_{\frac{4\pi}{3}} M R_{\frac{4\pi}{3}}^{-1}$	$R_{\frac{2\pi}{3}} M R_{\frac{2\pi}{3}}^{-1}$	M	$R_{\frac{2\pi}{3}}$	$R_{\frac{4\pi}{3}}$
Character $\chi_S(\sigma) = tr(S_\sigma)$	1	-1	-1	-1	1	1

For further discussion in class, we collect these representations into a table, along with the trivial representation:

Group element σ	identity	(AB)	(AC)	(BC)	(ABC)	(ACB)
Permutation matrix U_σ	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$
Permutation rep: $\chi_U(\sigma)$	3	1	1	1	0	0
2-d rep from G: $\chi_L(\sigma)$	2	0	0	0	-1	-1
1-d rep from G: $\chi_S(\sigma)$	1	-1	-1	-1	1	1
trivial rep: $\chi_I(\sigma)$	1	1	1	1	1	1

Note that $\chi_U = \chi_L + \chi_I$. Note also that $\sum_{\sigma\in P} |\chi_L(\sigma)|^2 = \sum_{\sigma\in P} |\chi_S(\sigma)|^2 = \sum_{\sigma\in P} |\chi_I(\sigma)|^2 = 6$ but that $\sum_{\sigma\in P} |\chi_U(\sigma)|^2 = 12$. And note that $\sum_{\sigma\in P} \chi_L(\sigma)\overline{\chi_S(\sigma)} = \sum_{\sigma\in P} \chi_L(\sigma)\overline{\chi_I(\sigma)} = \sum_{\sigma\in P} \chi_S(\sigma)\overline{\chi_I(\sigma)} = 0$.