Linear Transformations and Group Representations

Homework #3 (2018-2019), Answers

Q1: Complete the character table of the group consisting of rotations and mirrorings of the cube.

rot90: rotation by 90 deg around any face
rot180: rotation by 180 deg around any face
edge flip: rotation by 180 deg around the line that connects the midpoints of two opposite edges
corner rot 120: rotation by 120 deg around the line that connects opposite corners
invert indicates that the above operation is followed by inversion through the center of the cube

Only the named representations are irreducible; the representations in red are NOT irreducible are starting points from which we remove a previously-determined irreducible representation by subtracting its character

element description	ident.	rot 90	rot 180	edge flip	corner rot 120	invert	rot 90 invert	rot 180 invert	edge flip invert	corner rot 120 invert	sum of squares
element count	1	6	3	6	8	1	6	3	6	8	
representation	**Character**										
trivial	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	48
	Below: representation based on parity of the number of inversions										
parity	+1	+1	+1	+1	+1	-1	-1	-1	-1	-1	48
	Below: representation based on permutations of the three opposite face pairs										
	+3	+1	+3	+1	0	+3	+1	+3	+1	0	96
	Below: remove the projection on the trivial										
face-pair	+2	0	+2	0	-1	+2	0	+2	0	-1	48
	Below: tensor product face-pair with parity										
face-pair ⊗ parity	+2	0	+2	0	-1	-2	0	-2	0	+1	48
	Below: representation based on 3D rotation matrices										
3D	+3	+1	-1	-1	0	-3	-1	+1	+1	0	48
	Below: tensor product 3D with parity										
3D ⊗ parity	+3	+1	-1	-1	0	+3	+1	-1	-1	0	48
	Below: representation based on permutations of the four main diagonals										
	+4	0	0	+2	+1	+4	0	0	+2	+1	96
	Below: remove the projection on the trivial										
maindiag	+3	-1	-1	+1	0	+3	-1	-1	+1	0	48
	Below: tensor product maindiag with parity										
maindiag ⊗ parity	+3	-1	-1	+1	0	-3	+1	+1	-1	0	48
	Below: representation based parity of permutations of the four main diagonals										
MDparity	+1	-1	+1	-1	+1	+1	-1	+1	-1	+1	48
	Below: representation based parity of permutations of the four main diagonals										
MDparity ⊗ parity	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	48

"parity" comes from the number of inversions through the origin, i.e., whether the action turns keeps handedness (preserves parity) or does not (inverts parity).
"MDparity" comes from first considering the transformations of the cube to be permutations on the four main diagonals, and then determining the parity of that permutation (as an even or an odd permutation)
Sum of squares should be 48, the size of the group, for irreducible representations.
You can check that dot-products of any pair of irreducible rows are zero.

Q2: Consider the same group as in Q1 as a permutation group on the 8 vertices. Compute the character of this representation, and determine if it is irreducible. If not, determine its decomposition into irreducible representations.

element description	ident.	rot 90	rot 180	edge flip	corner rot 120	invert	rot 90 invert	rot 180 invert	edge flip invert	corner rot 120 invert	sum of squares
element count	1	6	3	6	8	1	6	3	6	8	
representation	**Character**										
	Below: representation based on permutations of the eight vertices										
vertex	+8	0	0	0	+2	0	0	0	4	0	192

The sum of squares is $1 \bullet 8^2 + 8 \bullet 2^2 + 6 \bullet 4^2 = 64 + 32 + 96 = 192 = 4|G|$, so this representation is reducible. Since it squared-length is 4, we expect that it will either (a) direct sum of two copies of the same irreducible representation, or (b) be a direct sum of four different irreducible representations. But also, its dimension is 8, which must be the total dimension of the component representations. Since the irreducible representations of G have dimensions 1, 2, and 3, the only possibility is that the vertex representation is the direct sum of two different 1-dimensoinal irreducible representations, and two different 3-dimensional irreducible representations.

To find them, we compute the dot-product (normalized by $|G| = 48$) of the above character with each of the characters computed in Q1. The following yield a value of 1

element description	ident.	rot 90	rot 180	edge flip	corner rot 120	invert	rot 90 invert	rot 180 invert	edge flip invert	corner rot 120 invert	dot prod
element count	1	6	3	6	8	1	6	3	6	8	
representation	**Character**										
	Below: representation based on permutations of the eight vertices										
vertex	+8	0	0	0	+2	0	0	0	+4	0	192
trivial	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	48
MDparity $\otimes$ parity	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	48
3D	+3	+1	-1	-1	0	-3	-1	+1	+1	0	48
maindiag	+3	-1	-1	+1	0	+3	-1	-1	+1	0	48

As a check, note that the four irreducible characters sum to the vertex character