Exam, 2018-2019 Questions and Solutions

Note that many of the answers are (much) more detailed than is required for "full credit"

1. Projection onto one-dimensional representations

1: 12 points total, A: 6, B, 6; dependencies: A->B

A. Projecting onto a space in which a one-dimensional representation acts.

Setup: Given a unitary representation (not necessarily irreducible) L of a finite group G in the vector space V ,we showed in class that $P_L(v)=\frac{1}{|G|}\sum_g L_g(v)$ is a projection of the group onto a subspace in which L acts as the trivial representation, i.e., that $L_g(P_L(v))=P_L(v)$ for all vectors $v\in V$.

Consider a one-dimensional non-trivial representation M , and define $Q_{M,L}(v)=\frac{1}{|G|}\sum_g \overline{\chi_M(g)}L_g(v)$.

A. Show that $Q_{M,L}(v)$ is a projection.

To show that that $Q_{M,L}(v)$ is a projection, we need to show that $Q_{M,L}Q_{M,L}=Q_{M,L}$.

$$Q_{M,L}\left(Q_{M,L}(v)\right)=\frac{1}{|G|}\sum_g \overline{\chi_M(g)}L_g\left(Q_{M,L}(v)\right)=\frac{1}{|G|}\sum_g \overline{\chi_M(g)}L_g\left(\frac{1}{|G|}\sum_h \overline{\chi_M(h)}L_h(v)\right).$$

Since L_g is linear, we can expand the outer application of it:

$$\frac{1}{|G|}\sum_g \overline{\chi_M(g)}L_g\left(\frac{1}{|G|}\sum_h \overline{\chi_M(h)}L_h(v)\right)=\frac{1}{|G|^2}\sum_g \overline{\chi_M(g)}\left(\sum_h L_g\left(\overline{\chi_M(h)}L_h(v)\right)\right)$$
$$=\frac{1}{|G|^2}\sum_g \overline{\chi_M(g)}\,\overline{\chi_M(h)}\left(\sum_h L_g\left(L_h(v)\right)\right).$$

Since M is one-dimensional (i.e., each M_g is one-dimensional), $\chi_M(g)=tr\left(M_g\right)=M_g$. Therefore $M_gM_h=M_{gh}$ (the defining property of a group representation) can be rewritten as $\chi_M(g)\chi_M(h)=\chi_M(gh)$.

$$\frac{1}{|G|^2}\sum_g \overline{\chi_M(g)}\,\overline{\chi_M(h)}\left(\sum_h L_g\left(L_h(v)\right)\right)=\frac{1}{|G|^2}\sum_{g,h} \overline{\chi_M(gh)}L_g\left(L_h(v)\right)=\frac{1}{|G|^2}\sum_{g,h} \overline{\chi_M(gh)}L_{gh}(v),$$ where the last equality uses the fact that L is a group representation, and the summations are over all pairs of g and h in G But as g and h range over all pairs of values in G , their product gh attains all values in

G, and attains each value exactly $|G|$ times. (This is because the multiplication table of G contains each element once.) So

$$\frac{1}{|G|^2}\sum_{g,h}\overline{\chi_M(gh)}L_{gh}(v)=\frac{1}{|G|^2}|G|\sum_r\overline{\chi_M(r)}L_r(v)=\frac{1}{|G|}\sum_r\overline{\chi_M(r)}L_r(v)=Q_{M,L}(v).$$

B. Show that if $w=Q_{M,L}(v)$, then $L_g(w)=M_g(w)$, i.e., that $Q_{M,L}(v)$ is a projection onto the subspace in which L acts like M.

$L_g(w)=L_g\left(Q_{M,L}(v)\right)=L_g\left(\frac{1}{|G|}\sum_h\overline{\chi_M(h)}L_h(v)\right)=\frac{1}{|G|}\left(\sum_h\overline{\chi_M(h)}L_gL_h(v)\right)=\frac{1}{|G|}\left(\sum_h\overline{\chi_M(h)}L_{gh}(v)\right)$, where the first equality uses the definition of w, the second equality uses the definition of $Q_{M,L}$, the third equality uses linearity of L_g, and the fourth equality uses the group representation property of L.

Now substitute $r=gh$, so $h=g^{-1}r$. As h runs over all elements of G, so does $r=gh$. So

$$\frac{1}{|G|}\left(\sum_h\overline{\chi_M(h)}L_{gh}(v)\right)=\frac{1}{|G|}\left(\sum_r\overline{\chi_M(g^{-1}r)}L_r(v)\right)=\frac{1}{|G|}\left(\sum_r\overline{\chi_M(g^{-1})}\,\overline{\chi_M(r)}L_r(v)\right).$$

As in part A, $\chi_M(g)\chi_M(h)=\chi_M(gh)$, so $\overline{\chi_M(g^{-1}r)}=\overline{\chi_M(g^{-1})}\,\overline{\chi_M(r)}$, leading to the last equality.

Also, because M_g is one-dimensional and also unitary (so it simply multiplication by its character, which is a complex number of magnitude 1), $\overline{\chi_M(g^{-1})}=\overline{M}_{g^{-1}}=\overline{M_g^{-1}}=M_g=\chi_M(g)$.

Finally,

$$\frac{1}{|G|}\left(\sum_r\overline{\chi_M(g^{-1})}\,\overline{\chi_M(r)}L_r(v)\right)=\chi_M(g)\frac{1}{|G|}\left(\sum_r\overline{\chi_M(r)}L_r(v)\right)=\chi_M(g)Q_{M,L}(v)=\chi_M(g)w=M_g(w)$$

So $L_g(w)=M_g(w)$.

2. Convolution on a group

2: 12 points total, A: 6; B: 3, C: 3; dependencies: A->B, A->C

This generalizes the notion of convolution of functions of a real variable (i.e,. on a line) to functions on a group.

A. Let L be a unitary representation on a group. Let b and c be two functions on the group. Define their convolution $y = b * c$ as a third function on the group, where $y(g) = \sum_h b(gh^{-1})c(h)$. (This should look just like convolution on the line, where the group operation corresponds to translation.) Define $\tilde{B}(L) = \sum_g b(g)L_g$, $\tilde{C}(L) = \sum_g c(g)L_g$, and $\tilde{Y}(L) = \sum_g y(g)L_g$. Show that $\tilde{Y}(L) = \tilde{B}(L)\tilde{C}(L)$.

$\tilde{Y}(L) = \sum_g y(g)L_g = \sum_g (b*c)(g)L_g = \sum_g \sum_h b(gh^{-1})c(h)L_g = \sum_g \sum_h b(gh^{-1})c(h)L_{gh^{-1}}L_h$.where the first three equalities are the given definitions and the last equality follows because L is structure-preserving, i.e., $L_g = L_{gh^{-1}h} = L_{gh^{-1}}L_h$. We reverse the order of summation (everything is finite), and, for the inner sum, which is now over g, change variables to $g = rh$. As g runs over the group, so does $r = gh^{-1}$. Therefore,

$$\tilde{Y}(L) = \sum_g \sum_h b(gh^{-1})c(h)L_{gh^{-1}}L_h = \sum_h \left(\sum_g b(gh^{-1})L_{gh^{-1}} \right) c(h)L_h = \sum_h \left(\sum_r b(r)L_r \right) c(h)L_h$$
$$= \sum_h \tilde{B}(L)c(h)L_h = \tilde{B}(L)\sum_h c(h)L_h = \tilde{B}(L)\tilde{C}(L).$$

Note: This result can be summarized by stating that the mapping $\tilde{B}(L) = \sum_g b(g)L_g$ between functions on the group and a stack of matrices $\tilde{B}(L)$ is a homomorphism from the “convolution algebra” on the group, to matrix multiplication.

B. Is convolution on a group commutative? Why or why not?

Not commutative unless the group is commutative, since for representations of 2 dimensions or more, $\tilde{B}(L)\tilde{C}(L) \neq \tilde{C}(L)\tilde{B}(L)$. Specifically: let $b(g) = \begin{cases} 1, & g = \beta \\ 0, & otherwise \end{cases}$ and $c(g) = \begin{cases} 1, & g = \gamma \\ 0, & otherwise \end{cases}$. Then $y(g) = \left(b*c\right)(g)$ is equal to 1 at $g = \beta\gamma$ and 0 otherwise (since we need both $gh^{-1} = \beta$ and $h = \gamma$ to get a non-zero contribution to $y(g) = \sum_h b(gh^{-1})c(h)$). On the other hand, $z(g) = \left(c*b\right)(g)$ is equal to 1 at $g = \gamma\beta$, and 0 otherwise. So for any group elements β and γ for which $\beta\gamma \neq \gamma\beta$, then $b*c \neq c*b$.

C. Is convolution on a group associative? Why or why not?

Yes. Consider $y=(a*b)*c$ and $z=a*(b*c)$. Using part A, we can compute $\tilde{Y}(L)=\left(\tilde{A}(L)\tilde{B}(L)\right)\tilde{C}(L)=\tilde{A}(L)\left(\tilde{B}(L)\tilde{C}(L)\right)=\tilde{Z}(L)$. Since this holds for all irreducible representations, and their matrix elements are an orthonormal basis for functions on the group, it follows that $y=z$

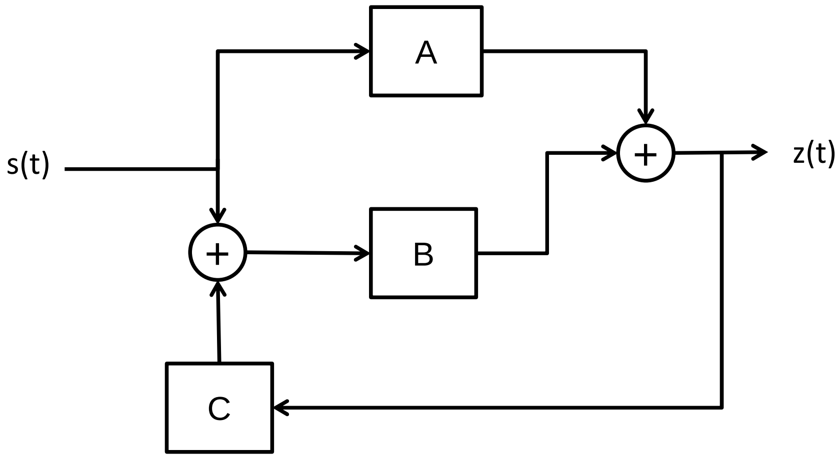
Alternative method. One can write the convolution $x=a*b$, where $x(g)=\sum_{h} a(gh^{-1})b(h)$, in the form $x(g)=\sum_{h_1,h_2} a(h_1)b(h_2)$, where the sum is over all h_1 and h_2 for which $h_1h_2=g$. Using this form, $y=(a*b)*c$ is equivalent to $y(g)=\sum_{h_1,h_2,h_3}\left(a(h_1)b(h_2)\right)c(h_3)$, where the sum is over all h_1, h_2 and h_3 for which $(h_1h_2)h_3=g$. Analogously, $z(g)=\sum_{h_1,h_2,h_3} a(h_1)\left(b(h_2)c(h_3)\right)$, where the sum is over all h_1, h_2 and h_3 for which $h_1(h_2h_3)=g$. These last two sums are identical, because of associativity of the group operation (so $(h_1h_2)h_3=g$ is equivalent to $h_1(h_2h_3)=g$), and associativity of multiplication (so $\left(a(h_1)b(h_2)\right)c(h_3)=a(h_1)\left(b(h_2)c(h_3)\right)$.

Note that the convolution is not a group homomorphism, as it is not a mapping from groups to groups, but rather, a mapping from a pair of functions on the group to a third function on the group.

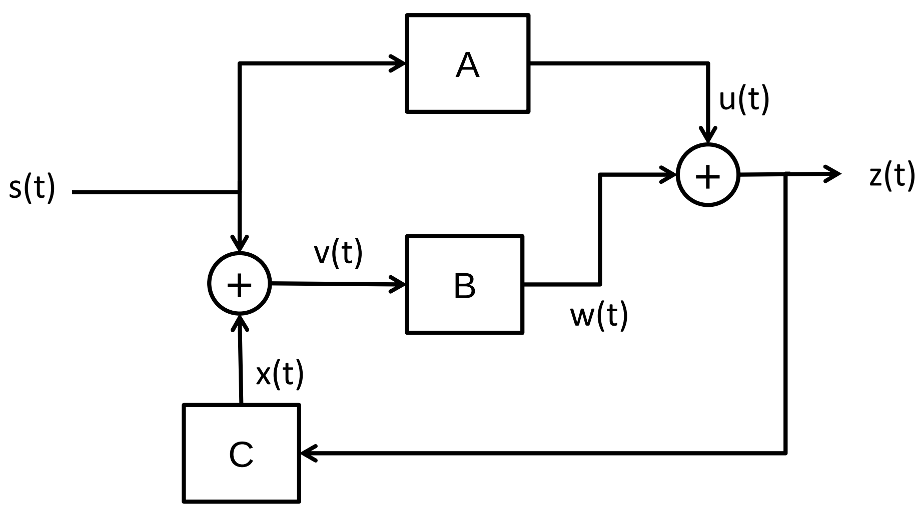
3. Linear systems problem

3. 10 points total, A: 4, B: 3, C: 3; dependencies: A->B->C

A. Consider the system block-diagram below,with input $s(t)$, *output* $z(t)$, *and linear filters* A, B, *and* C *with transfer functions* $\tilde{A}(\omega)$, $\tilde{B}(\omega)$, *and* $\tilde{C}(\omega)$. *Find the transfer function* $\tilde{Z}(\omega)$ *that relates* $\tilde{s}(\omega)$ *and* $\tilde{z}(\omega)$ *via* $\tilde{z}(\omega)=\tilde{Z}(\omega)\tilde{s}(\omega)$.



Chasing signals, as per diagram below:



$\tilde{u}(\omega)=\tilde{A}(\omega)\tilde{s}(\omega)$.

$\tilde{z}(\omega)=\tilde{u}(\omega)+\tilde{w}(\omega)=\tilde{A}(\omega)\tilde{s}(\omega)+\tilde{w}(\omega)$.

$\tilde{x}(\omega)=\tilde{C}(\omega)\tilde{z}(\omega)=\tilde{C}(\omega)\left(\tilde{A}(\omega)\tilde{s}(\omega)+\tilde{w}(\omega)\right)$

$\tilde{v}(\omega)=\tilde{s}(\omega)+\tilde{x}(\omega)=\tilde{s}(\omega)+\tilde{C}(\omega)\left(\tilde{A}(\omega)\tilde{s}(\omega)+\tilde{w}(\omega)\right)=\tilde{C}(\omega)\tilde{w}(\omega)+\left(1+\tilde{A}(\omega)\tilde{C}(\omega)\right)\tilde{s}(\omega)$

$\tilde{w}(\omega)=\tilde{B}(\omega)\tilde{v}(\omega)=\tilde{B}(\omega)\tilde{C}(\omega)\tilde{w}(\omega)+\tilde{B}(\omega)\left(1+\tilde{A}(\omega)\tilde{C}(\omega)\right)\tilde{s}(\omega)$, so

$\tilde{w}(\omega) = \dfrac{\tilde{B}(\omega)\left(1+\tilde{A}(\omega)\tilde{C}(\omega)\right)}{1-\tilde{B}(\omega)\tilde{C}(\omega)}\tilde{s}(\omega)$. Since $\tilde{z}(\omega) = \tilde{A}(\omega)\tilde{s}(\omega) + \tilde{w}(\omega)$ (above),

$$\tilde{z}(\omega) = \tilde{A}(\omega)\tilde{s}(\omega) + \frac{\tilde{B}(\omega)\left(1+\tilde{A}(\omega)\tilde{C}(\omega)\right)}{1-\tilde{B}(\omega)\tilde{C}(\omega)}\tilde{s}(\omega) = \frac{\tilde{A}(\omega)+\tilde{B}(\omega)}{1-\tilde{B}(\omega)\tilde{C}(\omega)}\tilde{s}(\omega)$$

So $\tilde{z}(\omega) = \tilde{Z}(\omega)\tilde{s}(\omega)$, where $\tilde{Z}(\omega) = \dfrac{\tilde{A}(\omega)+\tilde{B}(\omega)}{1-\tilde{B}(\omega)\tilde{C}(\omega)}$

B. Consider the special case of the above diagram, where A is multiplication by the constant a, B is a linear filter with impulse response $\frac{1}{\tau}e^{-t/\tau}$, and C is zero. Determine $\tilde{Z}(\omega)$.

$\tilde{A}(\omega) = a$.

$\tilde{B}(\omega) = \int\limits_0^\infty e^{-i\omega t}\left(\frac{1}{\tau}e^{-t/\tau}\right)dt = \frac{1}{\tau}\int\limits_0^\infty e^{-(i\omega+1/\tau)t}dt = \frac{1}{\tau}\frac{1}{i\omega+1/\tau}\int\limits_0^\infty e^{-u}du = \frac{1}{\tau}\frac{1}{i\omega+1/\tau} = \frac{1}{1+i\omega\tau}$, were we used the substitution $u = (i\omega+1/\tau)t$.

$\tilde{C}(\omega) = 0$. With $\tilde{Z}(\omega) = \dfrac{\tilde{A}(\omega)+\tilde{B}(\omega)}{1-\tilde{B}(\omega)\tilde{C}(\omega)}$ from part A, $\tilde{Z}(\omega) = a + \dfrac{1}{1+i\omega\tau} = \dfrac{a+1+ai\omega\tau}{1+i\omega\tau}$.

C. Consider the further special case of $a = -2$. Determine the amplitude of the response to a sinusoid $e^{i\omega t}$. Determine the effect of this transformation on the power spectrum.

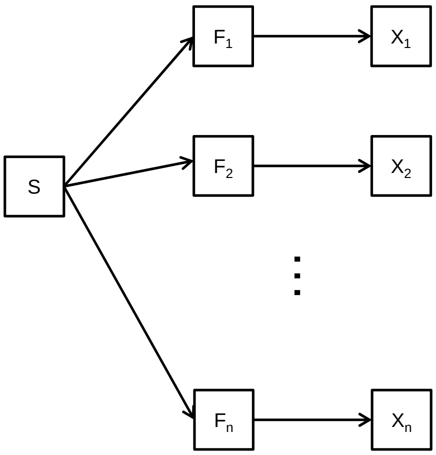
The amplitude of the response to a unit sinusoid given by $\left|\tilde{Z}(\omega)\right|$. $\left|\tilde{Z}(\omega)\right| = \left|\dfrac{-1-2i\omega\tau}{1+i\omega\tau}\right|$. The power spectrum of $s(t)$ and $z(t)$ are related by $P_Z(\omega) = \left|\tilde{Z}(\omega)\right|^2 P_S(\omega)$.

Note: What I meant to ask was, $a = -1/2$. In this case, $\left|\tilde{Z}(\omega)\right| = \left|\dfrac{1/2 - i\omega\tau/2}{1+i\omega\tau}\right| = \dfrac{1}{2}\left|\dfrac{1-i\omega\tau}{1+i\omega\tau}\right| = \dfrac{1}{2}$, so the power spectrum is reduced by a factor of 4.

4. Cross-spectra

4. 16 points total, A: 3, B: 2, C: 1; D: 2; E: 3, F: 2: G: 1, H: 2; dependencies: A->B->C->D, A->E->F->G, (A,E)->H

A. Consider a noise source $s(t)$ *with power spectrum* $P_S(\omega)$*, that is observed through* n *linear filters* F_i *with transfer functions* $\tilde{F}_i(\omega)$ *to generate signals* $x_i(t)$*. Compute the cross-spectra* $P_{X_i X_j}(\omega)$*.*



parts A-D

Let $\tilde{x}_i(\omega)$ be a Fourier estimate of $x_i(t)$ over some finite but long interval *T*, i.e., $\tilde{x}_i(\omega)=\int_0^T e^{-i\omega t}x_i(t)dt$, and similarly for $\tilde{s}(\omega)$. We want to compute $P_{X_i X_j}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{x}_i(\omega)\overline{\tilde{x}_j(\omega)}\right\rangle$, knowing $P_S(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{s}(\omega)\overline{\tilde{s}(\omega)}\right\rangle$. Since $\tilde{x}_i(\omega)=\tilde{F}_i(\omega)\tilde{s}(\omega)$, $\tilde{x}_i(\omega)\overline{\tilde{x}_j(\omega)}=\tilde{F}_i(\omega)\tilde{s}(\omega)\overline{\tilde{F}_j(\omega)\tilde{s}(\omega)}$ and

$$\begin{aligned}
P_{X_i X_j}(\omega) &= \lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{x}_i(\omega)\overline{\tilde{x}_j(\omega)}\right\rangle \\
&= \lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{F}_i(\omega)\tilde{s}(\omega)\overline{\tilde{F}_j(\omega)\tilde{s}(\omega)}\right\rangle \\
&= \tilde{F}_i(\omega)\overline{\tilde{F}_j(\omega)}\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{s}(\omega)\overline{\tilde{s}(\omega)}\right\rangle \\
&= P_S(\omega)\tilde{F}_i(\omega)\overline{\tilde{F}_j(\omega)}
\end{aligned}.$$

B. Consider the cross-spectra $P_{X_i X_j}(\omega)$ *as a matrix* $Z(\omega)$*. How many nonzero eigenvalues does it have? (i.e., what is its rank?)*

From A.

$$Z(\omega)=\begin{pmatrix} P_{X_1X_1} & \cdots & P_{X_1X_n} \\ \vdots & & \vdots \\ P_{X_nX_1} & \cdots & P_{X_nX_n} \end{pmatrix}=P_S(\omega)\begin{pmatrix} \tilde{F}_1(\omega) \\ \vdots \\ \tilde{F}_n(\omega) \end{pmatrix}\begin{pmatrix} \overline{\tilde{F}_1(\omega)} & \cdots & \overline{\tilde{F}_n(\omega)} \end{pmatrix}.$$

So, any product $Z\vec{v}$ is proportional to $\begin{pmatrix} \tilde{F}_1(\omega) \\ \vdots \\ \tilde{F}_n(\omega) \end{pmatrix}$, i.e., $Z(\omega)$ maps into a space of dimension 1. So it has rank 1.

C. Is $Z(\omega)$ self-adjoint?

Yes – for example, because $P_{X_iX_j}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{x}_i(\omega)\overline{\tilde{x}_j(\omega)}\right\rangle=\lim_{T\to\infty}\frac{1}{T}\overline{\left\langle \tilde{x}_j(\omega)\overline{\tilde{x}_i(\omega)}\right\rangle}=\overline{P_{X_iX_j}(\omega)}$.

D. Find a nonzero eigenvalue of $Z(\omega)$ and its associated eigenvector.

Since $Z(\omega)$ maps all into a space of dimension 1Its only nonzero eigenvector is $\vec{w}=\begin{pmatrix} \tilde{F}_1(\omega) \\ \vdots \\ \tilde{F}_n(\omega) \end{pmatrix}$.

Note: the nonzero eigenvalue can be found by

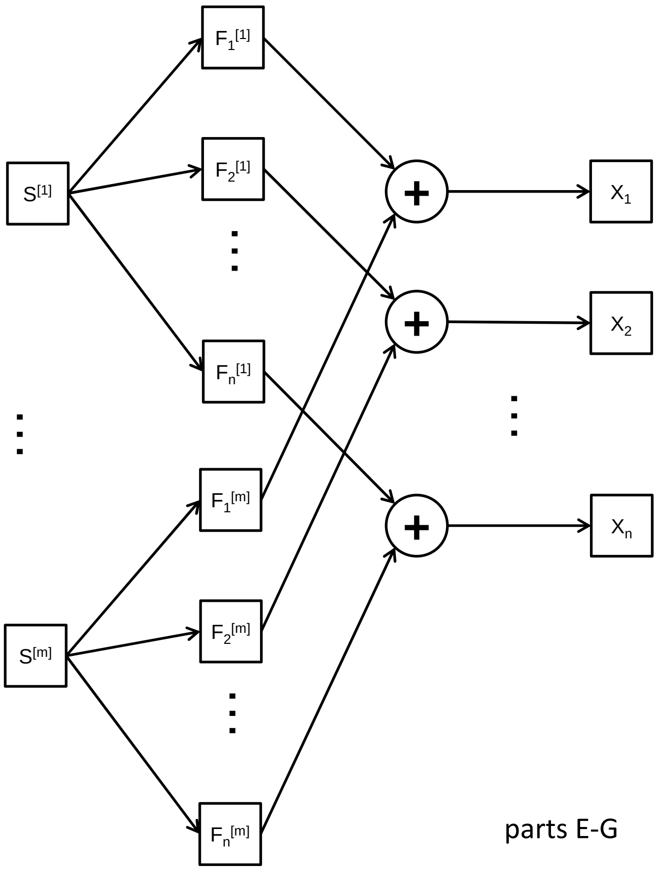
$$Z(\omega)\vec{u}=P_S(\omega)\begin{pmatrix} \tilde{F}_1(\omega) \\ \vdots \\ \tilde{F}_n(\omega) \end{pmatrix}\begin{pmatrix} \overline{\tilde{F}_1(\omega)} & \cdots & \overline{\tilde{F}_n(\omega)} \end{pmatrix}\begin{pmatrix} \tilde{F}_1(\omega) \\ \vdots \\ \tilde{F}_n(\omega) \end{pmatrix}=\begin{pmatrix} \tilde{F}_1(\omega) \\ \vdots \\ \tilde{F}_n(\omega) \end{pmatrix}P_S(\omega)\begin{pmatrix} \overline{\tilde{F}_1(\omega)} & \cdots & \overline{\tilde{F}_n(\omega)} \end{pmatrix}\begin{pmatrix} \tilde{F}_1(\omega) \\ \vdots \\ \tilde{F}_n(\omega) \end{pmatrix}$$

$$=\vec{u}P_S(\omega)\begin{pmatrix} \overline{\tilde{F}_1(\omega)} & \cdots & \overline{\tilde{F}_n(\omega)} \end{pmatrix}\begin{pmatrix} \tilde{F}_1(\omega) \\ \vdots \\ \tilde{F}_n(\omega) \end{pmatrix}=\vec{u}P_S(\omega)\sum_i\left|\tilde{F}_i(\omega)\right|^2$$

, so the nonzero eigenvalue is $P_S(\omega)\sum_i\left|\tilde{F}_i(\omega)\right|^2$.

Alternatively, since we know that there is only one nonzero eigenvalue, it is equal to the trace of $Z(\omega)$, which is the above expression.

E. Now consider m noise sources $s^{[k]}(t)$, uncorrelated, each with power spectrum $P_S^{[k]}(\omega)$, that these are observed as n signals $x_i(t)$, with the input of the k th source to $x_i(t)$ passing through the filter $F_i^{[k]}$ (diagram below). Assume that $F_i^{[k]}$ has transfer function $\tilde{F}_i^{[k]}(\omega)$, and that these m inputs are combined additively to create the signal $x_i(t)$. Compute the cross-spectra $P_{X_iX_j}(\omega)$.



parts E-G

Each signal $x_i(t)$ is a sum of signals from each noise source, i.e., $\tilde{x}_i(\omega) = \sum_k \tilde{F}_i^{[k]}(\omega)\tilde{s}^{[k]}(\omega)$. As in (a), $\tilde{x}_i(\omega)\overline{\tilde{x}_j(\omega)} = \sum_{k=1}^{m}\sum_{l=1}^{m} \tilde{F}_i^{[k]}(\omega)\tilde{s}^{[k]}(\omega)\overline{\tilde{F}_j^{[l]}(\omega)\tilde{s}^{[l]}(\omega)} = \sum_{k=1}^{m}\sum_{l=1}^{m} \tilde{F}_i^{[k]}(\omega)\overline{\tilde{F}_j^{[l]}(\omega)}\tilde{s}^{[k]}(\omega)\overline{\tilde{s}^{[l]}(\omega)}$, Since the noise sources are independent, the mixed terms ($k \neq l$) vanish when we take expectations:

$$P_{X_i X_j}(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{x}_i(\omega)\overline{\tilde{x}_j(\omega)} \right\rangle = \sum_{k=1}^{m}\sum_{l=1}^{m} \tilde{F}_i^{[k]}(\omega)\overline{\tilde{F}_j^{[l]}(\omega)}\left(\lim_{T\to\infty} \frac{1}{T}\left\langle \tilde{s}^{[k]}(\omega)\overline{\tilde{s}^{[l]}(\omega)} \right\rangle\right).$$

$$= \sum_{k=1}^{m} P_S^{[k]}(\omega)\tilde{F}_i^{[k]}(\omega)\overline{\tilde{F}_j^{[k]}(\omega)}$$

F. Consider the cross-spectra $P_{X_i X_j}(\omega)$ *as a matrix* $Z_m(\omega)$. *What is the maximum number of nonzero eigenvalues that it can have? (i.e., what is its maximum rank?)*

From E:

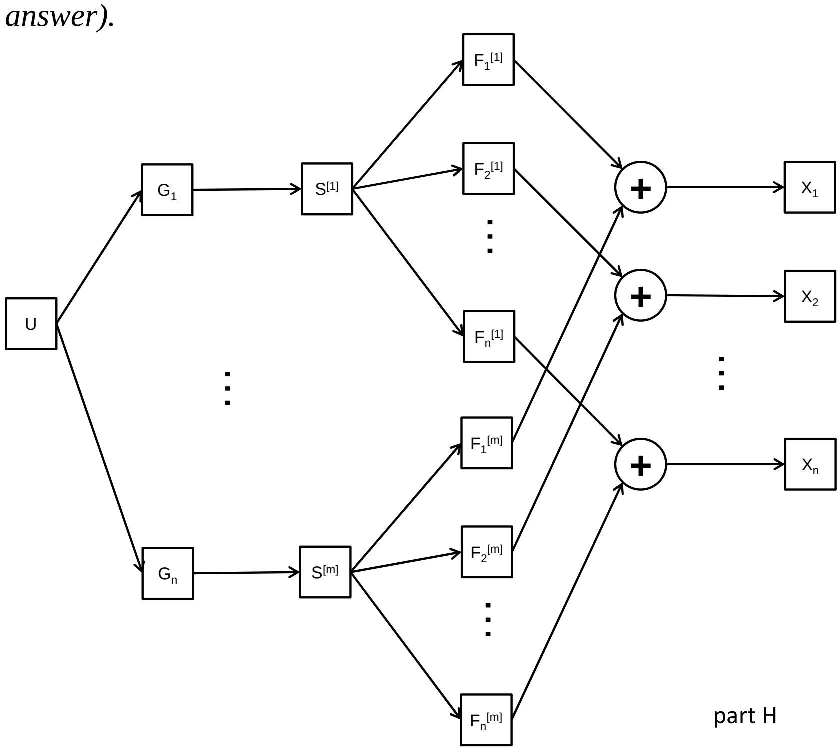
$Z_m(\omega) = \sum_{k=1}^{m} P_S^{[k]}(\omega) \begin{pmatrix} \tilde{F}_1^{[k]}(\omega) \\ \vdots \\ \tilde{F}_n^{[k]}(\omega) \end{pmatrix} \begin{pmatrix} \overline{\tilde{F}_1^{[k]}(\omega)} & \cdots & \overline{\tilde{F}_n^{[k]}(\omega)} \end{pmatrix}$. So any product $Z_m(\omega)\vec{v}$ is a linear combination of m vectors, $\begin{pmatrix} \tilde{F}_1^{[k]}(\omega) \\ \vdots \\ \tilde{F}_n^{[k]}(\omega) \end{pmatrix}$, for $k = 1,\ldots,m$. Maximum rank is therefore m, unless m exceeds the size of $Z_m(\omega)$. Maximum rank is therefore $\max(m,n)$.

Note: The ratio of the largest eigenvalue to the sum of the eigenvalues is called the global coherence.

G. Is $Z_m(\omega)$ self-adjoint?

Yes, it is a sum of terms of the form of C. Alternatively, note that power spectra are necessarily real, so $\overline{P_{X_i X_j}(\omega)} = \sum_{k=1}^{m} \overline{P_S^{[k]}(\omega)\tilde{F}_i^{[k]}(\omega)\overline{\tilde{F}_j^{[k]}(\omega)}} = \sum_{k=1}^{m} P_S^{[k]}(\omega)\overline{\tilde{F}_i^{[k]}(\omega)}\tilde{F}_j^{[k]}(\omega) = P_{X_j X_i}(\omega)$.

H. Now assume that the m noise sources $s^{[k]}(t)$ in E all arise from a single underlying signal $u(t)$, with $s^{[k]}(t)$ the result of filtering $u(t)$ by a linear filter G_k with transfer function $\tilde{G}_k(\omega)$. What is the maximum rank of the cross-spectral matrix $P_{X_i X_j}(\omega)$? (No need to derive this matrix, just justify the answer).



part H

Here, each output signal $x_i(t)$ is the result of filtering the same underlying noise $u(t)$ by a linear filter, albeit one that is more complicated than the one considered in A: its transfer function is $\sum_{k=1}^{m}\tilde{G}_k(\omega)\tilde{F}_i^{[k]}(\omega)$. So, as in A, the maximum rank of the cross-spectral matrix is 1.

5. Some properties of PCA

5. 16 points total, 4 points each subpart; no dependencies (A, B, C, D are parallel)

We start with the standard setup and formal solution for principal components analysis of a $n \times k$ *matrix* Y. *For definiteness, think of* Y *as an array of* k *snapshots of images, each of which has* n *pixels. As in the notes, the principal-components decomposition is given by* $Y = X\Lambda^{1/2}Z$ *where* X *is* $n \times p$ *with orthonormal columns,* Z *is* $p \times k$ *with orthonormal rows, and* Λ *is a diagonal matrix whose entries, all non-negative, are the eigenvalues of* YY^*.

As in the notes, one can either seek X as the first p (column) eigenvectors of the $n \times n$ *matrix* YY^* *and then find* $Z = \Lambda^{-1/2}X^*Y$, *or seek Z as the first p (row) eigenvectors of the* $k \times k$ *matrix* Y^*Y, *and then find* $X = YZ^*\Lambda^{-1/2}$.

Determine the effects of the following transformations on the principal-components decomposition, or, alternatively, whether the effects cannot be readily predicted; justify your answer.

A. Permuting the order of the snapshots
B. Permuting the order of the pixels
C. At each pixel, subtracting its mean throughout the dataset (i.e., subtracting the "average frame")
D. Replacing each pair of frames by their sum and their difference (assume an even number of frames)

In each of the answers below, let Y' be the modified dataset, and let X', Λ', and Z' be its principal-components decomposition, i.e., $Y' = X'(\Lambda')^{1/2}Z'$ with conditions on X', Λ', and Z' as for X, Λ, and Z.

A. Permuting the order of the snapshots corresponds to taking $Y' = YP$, where P is a $k \times k$ permutation matrix. $Y'(Y')^* = YP(YP)^* = YPP^*Y^* = YY^*$, with the final equality holding since P is a permutation matrix. Since $Y'(Y')^* = YY^*$, their eigenvalues and eigenvectors are the same, i.e., $\Lambda' = \Lambda$ and $X' = X$. $Z' = (\Lambda')^{-1/2}(X')^*Y' = \Lambda^{-1/2}X^*(YP) = ZP$, so Z' is a permutation of the columns of Z.

B. This is entirely parallel with A, but reverses the role of rows and columns. Permuting the order of the pixels corresponds to taking $Y' = PY$. We use the alternative formulation that Λ is a diagonal matrix whose entries are the eigenvalues of Y^*Y. $(Y')^*Y' = (PY)^*PY = Y^*P^*PY = Y^*Y$. Since $(Y')^*Y' = Y^*Y$, $\Lambda' = \Lambda$ and $Z' = Z$. $X' = Y'(Z')^*(\Lambda')^{-1/2} = PYZ^*\Lambda^{-1/2} = PX$, so X' is a permutation of the rows of X.

C. Subtracting the mean at each pixel across the movie. This corresponds to $Y' = YA$, where $A = I - \frac{1}{k}W$, where W is a $k \times k$ matrix consisting of all 1's. $Y'(Y')^* = YA(YA)^* = YAA^*Y^*$.

$AA^* = A^2 = \left(I - \frac{1}{k}W\right)\left(I - \frac{1}{k}W\right) = I - \frac{1}{k}W - \frac{1}{k}W + \frac{1}{k^2}W^2 = I - \frac{1}{k}W = A$. This is not a surprise; subtracting the mean of a movie that has already been mean-subtracted should not produce a change. However, there is no general relationship between $Y'(Y')^* = YAY^*$ and YY^*. One extreme: each pixel already has a mean of zero across the dataset. Then $Y' = Y$. Another extreme: all frames are identical. Then $Y' = 0$.

All that one can say is that $Y'(Y')^*$ typically has a rank that is one less than the rank of YY^*. This is because the columns of Y', i.e., the images, have a sum of zero, and hence, have a linear dependence. Algebraically, since $Y' = YA$, then $Y'(I - A) = YA(I - A) = Y(A - A^2) = Y(A - A) = 0$.

D. Pairwise sums and differences. This corresponds to $Y' = YM$, where M is the block-diagonal matrix $M = \begin{pmatrix} 1 & 1 & & & \\ 1 & -1 & & & \\ & & \ddots & & \\ & & & 1 & 1 \\ & & & 1 & -1 \end{pmatrix}$.

$Y'(Y')^* = YM(YM)^* = YMM^*Y^*$. Note that $MM^* = 2I$, so $Y'(Y')^* = 2YY^*$. So the eigenvalues of Λ' are twice those of Λ, and their eigenvectors are the same ($X' = X$).

$Z' = (\Lambda')^{-1/2}(X')^*Y' = \frac{1}{\sqrt{2}}\Lambda^{-1/2}X^*(YM) = \frac{1}{\sqrt{2}}ZM$, so Z' is a scaled version of the columns of Z, following pairwise addition and subtraction.

6. Group representations and graph Laplacians

6. 16 points total: A: 3, B: 3, C: 2, D: 2, E: 2, F: 2, G: 2; dependencies: A->B; C->D->E->F->G

Setup: a graph (bidirectional, unweighted) with an incidence matrix A, and a permutation that acts on the points of the graph, leaving the graph invariant, which we represent as a matrix P.

A. Show that P commutes with the graph Laplacian L.

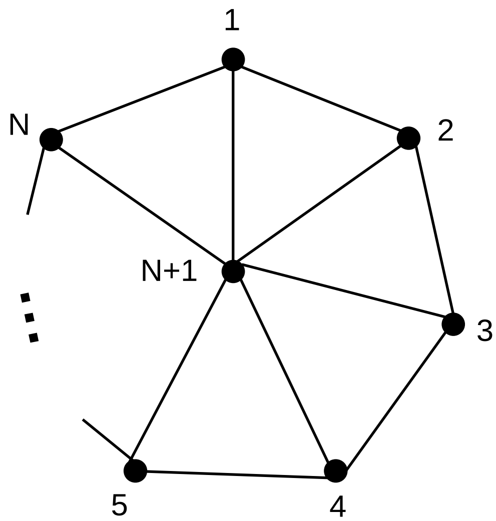
Method 1. The graph Laplacian is given by $L = D - A$, the difference of the diagonal matrix of the degrees (D) and the adjacency matrix. If the permutation P leaves the graph invariant, then $A = P^{-1}AP$. That is, nodes i and j are connected if, and only if, their images under the permutation are connected; the right-hand side is the adjacency matrix for the graph after transformation by P. So Note that a proper solution must make use of the hypothesis that P .*leaves the graph invariant, and cannot assume that P is symmetric.*
$PA = AP$. Similarly, the diagonal matrix of degrees of the transformed graph is $P^{-1}DP$. Since this is also invariant under the permutation (because it takes each node to a node of the same degree), $PD = DP$. So $LP = DP - AP = PD - PA = PL$.

Method 2. The graph Laplacian can also be defined as the linear operator that describes how functions on the graph "diffiuse:, i.e., $\frac{d}{dt}\vec{x} = -L\vec{x}$. Since P leaves the graph invariant, it leaves heat diffusion invariant. So $\frac{d}{dt}(P\vec{x}) = -L(P\vec{x})$ and $\frac{d}{dt}\vec{x} = -P^{-1}L(P\vec{x})$. Since this holds for all $\vec{x}$, $P^{-1}LP = L$, and $LP = PL$

B. Now let G be the group of all permutations that leave the graph invariant. There is a linear representation M of G in the space of functions on the graph: the permutation P corresponds to the linear transformation $\vec{x} \to P\vec{x}$, where $\vec{x}$ is a column vector of values assigned to each node. Say that this representation has an irreducible component M of dimension 1, and that $\vec{v}$ is a vector in this one-dimensional space. Show that $\vec{v}$ is an eigenvector of the graph Laplacian L.

Since $\vec{v}$ is a vector in the one-dimensional space in which M acts, $P\vec{v} = \chi_M(P)\vec{v}$ (where $\chi_M(P)$ is the scalar that M assigns to P). Conversely, if this relationship holds, then $\vec{v}$ is in the one-dimensional space corresponding to M. By part (a), $LP = PL$ for all $P \in G$. So $P(L\vec{v}) = L(P\vec{v}) = L\chi_M(P)\vec{v} = \chi_M(P)(L\vec{v})$. So M acts as scalar multiplication by $\chi_M(P)$ on $L\vec{v}$. Since this space is one-dimensional, $L\vec{v}$ is a scalar multiple of $\vec{v}$, so $\vec{v}$ is an eigenvector of L.

C. Consider the the "wagonwheel" graph with $N+1$ nodes (one in the center, N on the rim), $N \geq 3$. Write its graph Laplacian.



Let the first N nodes be on the rim, and the $N+1$th node be at the center. The nodes on the rim each have 3 neighbors (two neighbors on the rim, and the central node); the node at the center has N neighbors. So

$$L = D - A = \begin{pmatrix} 3 & -1 & & & -1 & -1 \\ -1 & 3 & -1 & & & -1 \\ & & \ddots & & & \vdots \\ & & -1 & 3 & -1 & -1 \\ -1 & & & -1 & 3 & -1 \\ -1 & -1 & \cdots & -1 & -1 & N \end{pmatrix}$$, i.e., the upper left $N \times N$ block has 3 on the diagonal, and -1 above and below the diagonal; the rightmost and bottom-most rows are-1, and the lower right element is $-N$.

D. The cyclic group on N elements, considered to act on the N rim nodes, leaves the graph invariant. Recall that the cyclic group has irreducible representations M_r , in which a rotation by $2\pi k / N$ acts like multiplication by $e^{2\pi ikr/N}$, for each $r = 0,1,...,N-1$. For each nonzero r, find a function $\vec{v}^{[r]}$ on the graph for which rotation by $2\pi k / N$ acts like multiplication by $e^{2\pi ikr/N}$. Show that, up to scalar multiples, there is only one such function.

Without loss of generality, we can assume that the first rim node is assigned a value 1 (since we only have to find $\vec{v}^{[r]}$ up to a multiplicative constant). A rotation by $2\pi(n-1)/N$ moves the first node to the position of the n th node. Therefore, at the n th node ($n \le N$), $\vec{v}$ has the value $e^{2\pi i(n-1)r/N}$. To determine the value of $\vec{v}^{[r]}$ at the central node, observe that this node is not changed by the permutation. So whatever value is assigned to the central node must be unchanged by multiplication by $e^{2\pi ikr/N}$, and therefore must be zero. So, $\vec{v}^{[r]} = \begin{pmatrix} 1 \\ e^{2\pi ir/N} \\ e^{4\pi ir/N} \\ \vdots \\ e^{(N-1)\pi i/N} \\ 0 \end{pmatrix}$. Since $\vec{v}^{[r]}$ was fully determined by our choice of its value at one node, there is only one possibility for $\vec{v}^{[r]}$ up to scalar multiple.

E. Find the eigenvalue of L corresponding to $\vec{v}^{[r]}$.

$$L\vec{v}^{[r]} = L\begin{pmatrix} 1 \\ e^{2\pi ir/N} \\ e^{4\pi ir/N} \\ \vdots \\ e^{(N-1)\pi i/N} \\ 0 \end{pmatrix} = \begin{pmatrix} -e^{-2\pi ir/N} + 3 - e^{2\pi ir/N} \\ -1 + 3e^{2\pi ir/N} - e^{4\pi ir/N} \\ -e^{2\pi ir/N} + 3e^{4\pi ir/N} - e^{6\pi ir/N} \\ \vdots \\ e^{(N-2)\pi i/N} + 3e^{(N-1)\pi i/N} - 1 \\ 0 \end{pmatrix} = \left(-e^{-2\pi ir/N} + 3 - e^{2\pi ir/N}\right)\vec{v}^{[r]}.$$ So the eigenvalue is $-e^{-2\pi ir/N} + 3 - e^{2\pi ir/N} = 2 - 2\cos\frac{2\pi r}{N}$.

F. In D and E, we found $N-1$ eigenvectors and eigenvalues, one for each of the $N-1$ nontrivial representations of the cyclic group. So there are two more eigenvectors and eigenvalues to find, and they both must correspond to the trivial representation. Find the functions on the graph for which the permutation group acts trivially (i.e, leaves the functions unchanged).

Assign the value a to the first node on the rim. Since rotations on the graph move this node to each of the other rim nodes, then all other rim nodes must be assigned the value a. Any value b can be assigned to the central node, and it will be invariant under rotation. So the vectors are $\vec{w}_{a,b} = \begin{pmatrix} a \\ a \\ \vdots \\ a \\ b \end{pmatrix}$.

G. Find two eigenvectors with distinct eigenvalues among the vectors identified in F.

$$L\vec{w}_{a,b} = L\begin{pmatrix} a \\ a \\ \vdots \\ a \\ b \end{pmatrix} = \begin{pmatrix} 3a-2a-b \\ 3a-2a-b \\ \vdots \\ 3a-2a-b \\ Nb-Na \end{pmatrix} = \begin{pmatrix} a-b \\ a-b \\ \vdots \\ a-b \\ N(b-a) \end{pmatrix} = \vec{w}_{a-b,N(b-a)}.$$ So the eigenvalues and eigenvectors satisfy the two equations $\begin{cases} a-b = \lambda a \\ N(b-a) = \lambda b \end{cases}$, which we may write as $\begin{pmatrix} 1 & -1 \\ -N & N \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$.

So if $\vec{w}_{a,b}$ is an eigenvector of L with eigenvalue λ, then $\begin{pmatrix} a \\ b \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 1 & -1 \\ -N & N \end{pmatrix}$ with λ. This matrix has characteristic equation $\det\begin{pmatrix} 1-z & -1 \\ -N & N-z \end{pmatrix} = 0$, which is $z^2 - (N+1)z = 0$. The eigenvalues are its roots, namely, 0 and $N+1$. Choosing $a = b$ yields the eigenvalue 0; choosing $a = 1$, $b = -N$ yields the eigenvalue $N+1$.