Exam, 2018-2019 Questions

Do any 40 points (or more).

Points and dependencies (dependencies are a guideline):
1: 12 points total, A: 6, B, 6; dependencies: A->B
2: 12 points total, A: 6; B: 3, C: 3; dependencies: A->B, A->C
3. 10 points total, A: 4, B: 3, C: 3; dependencies: A->B->C
4. 16 points total, A: 3, B: 2, C: 1; D: 2; E: 3, F: 2: G: 1, H: 2; dependencies: A->B->C->D, A->E->F->G, (A,E)->H
5. 16 points total, 4 points each subpart; no dependencies (A, B, C, D are parallel)
6. 16 points total: A: 3, B: 3, C: 2, D: 2, E: 2, F: 2, G: 2; dependencies: A->B; C->D->E->F->G

1. Projection onto one-dimensional representations

A. Projecting onto a space in which a one-dimensional representation acts.

Setup: Given a unitary representation (not necessarily irreducible) L of a finite group G in the vector space V ,we showed in class that $P_L(v) = \frac{1}{|G|}\sum_g L_g(v)$ is a projection of the group onto a subspace in which L acts as the trivial representation, i.e., that $L_g(P_L(v)) = P_L(v)$ for all vectors $v \in V$.

Consider a one-dimensional non-trivial representation M , and define $Q_{M,L}(v) = \frac{1}{|G|}\sum_g \overline{\chi_M(g)} L_g(v)$.

A. Show that $Q_{M,L}(v)$ is a projection.

B. Show that if $w = Q_{M,L}(v)$, then $L_g(w) = M_g(w)$, i.e., that $Q_{M,L}(v)$ is a projection onto the subspace in which L acts like M .

2. Convolution on a group

This generalizes the notion of convolution of functions of a real variable (i.e,. on a line) to functions on a group.

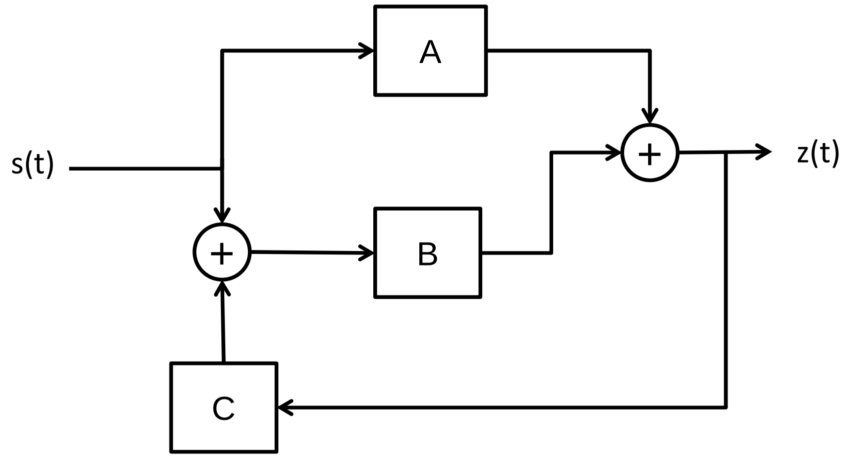
A. Let L be a unitary representation on a group. Let b and c be two functions on the group. Define their convolution $y = b * c$ as a third function on the group, where $y(g) = \sum_h b(gh^{-1})c(h)$. (This should look just like convolution on the line, where the group operation corresponds to translation.)
Define $\tilde{B}(L) = \sum_g b(g)L_g$, $\tilde{C}(L) = \sum_g c(g)L_g$, and $\tilde{Y}(L) = \sum_g y(g)L_g$. Show that $\tilde{Y}(L) = \tilde{B}(L)\tilde{C}(L)$.

B. Is convolution on a group commutative? Why or why not?

C. Is convolution on a group associative? Why or why not?

3. Linear systems problem

A. Consider the system block-diagram below,with input $s(t)$, output $z(t)$, and linear filters A, B, and C with transfer functions $\tilde{A}(\omega)$, $\tilde{B}(\omega)$, and $\tilde{C}(\omega)$. Find the transfer function $\tilde{Z}(\omega)$ that relates $\tilde{s}(\omega)$ and $\tilde{z}(\omega)$ via $\tilde{z}(\omega)=\tilde{Z}(\omega)\tilde{s}(\omega)$.



B. Consider the special case of the above diagram, where A is multiplication by the constant a, B is a linear filter with impulse response $\frac{1}{\tau}e^{-t/\tau}$, and C is zero. Determine $\tilde{Z}(\omega)$.

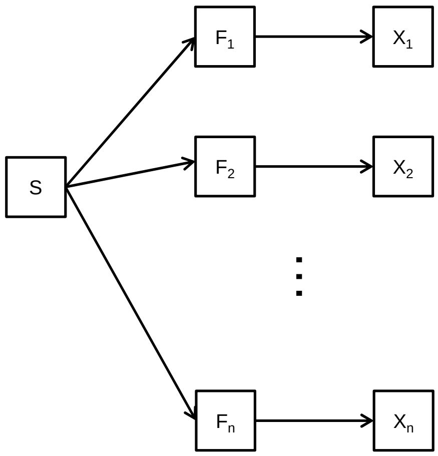
$\tilde{A}(\omega)=a$.

$\tilde{B}(\omega)=\int_0^\infty e^{-i\omega t}\left(\frac{1}{\tau}e^{-t/\tau}\right)dt=\frac{1}{\tau}\int_0^\infty e^{-(i\omega+1/\tau)t}dt=\frac{1}{\tau}\frac{1}{i\omega+1/\tau}\int_0^\infty e^{-u}du=\frac{1}{\tau}\frac{1}{i\omega+1/\tau}=\frac{1}{1+i\omega\tau}$, were we used the substitution $u=(i\omega+1/\tau)t$.

C. Consider the further special case of $a=-2$. Determine the amplitude of the response to a sinusoid $e^{i\omega t}$. Determine the effect of this transformation on the power spectrum.

4. Cross-spectra

A. Consider a noise source $s(t)$ with power spectrum $P_S(\omega)$, that is observed through n linear filters F_i with transfer functions $\tilde{F}_i(\omega)$ to generate signals $x_i(t)$. Compute the cross-spectra $P_{X_i X_j}(\omega)$.



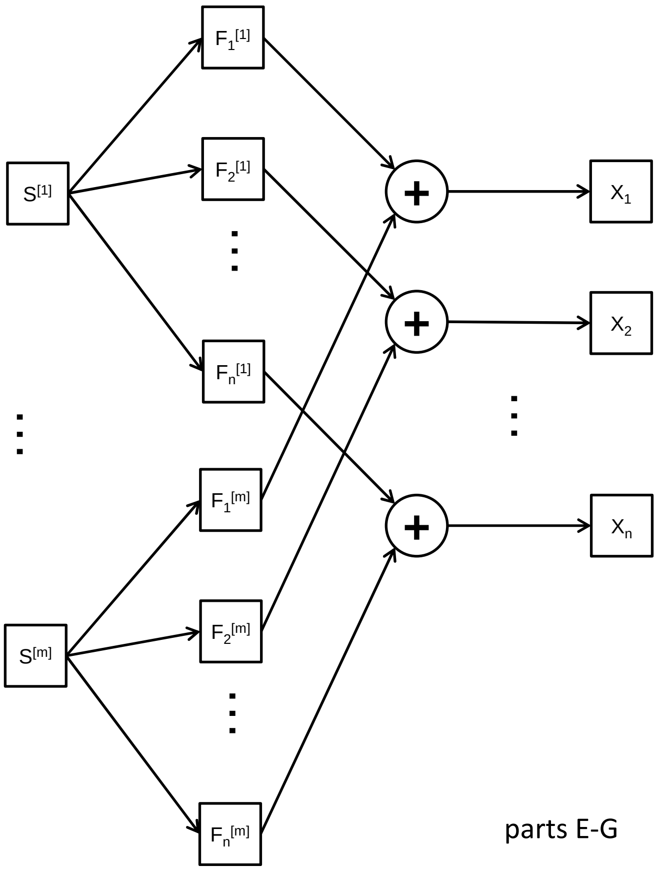
parts A-D

B. Consider the cross-spectra $P_{X_i X_j}(\omega)$ as a matrix $Z(\omega)$. How many nonzero eigenvalues does it have? (i.e., what is its rank?)

C. Is $Z(\omega)$ self-adjoint?

D. Find a nonzero eigenvalue of $Z(\omega)$ and its associated eigenvector.

E. Now consider m noise sources $s^{[k]}(t)$, uncorrelated, each with power spectrum $P_S^{[k]}(\omega)$, that these are observed as n signals $x_i(t)$, with the input of the k th source to $x_i(t)$ passing through the filter $F_i^{[k]}$ (diagram below). Assume that $F_i^{[k]}$ has transfer function $\tilde{F}_i^{[k]}(\omega)$, and that these m inputs are combined additively to create the signal $x_i(t)$. Compute the cross-spectra $P_{X_i X_j}(\omega)$.

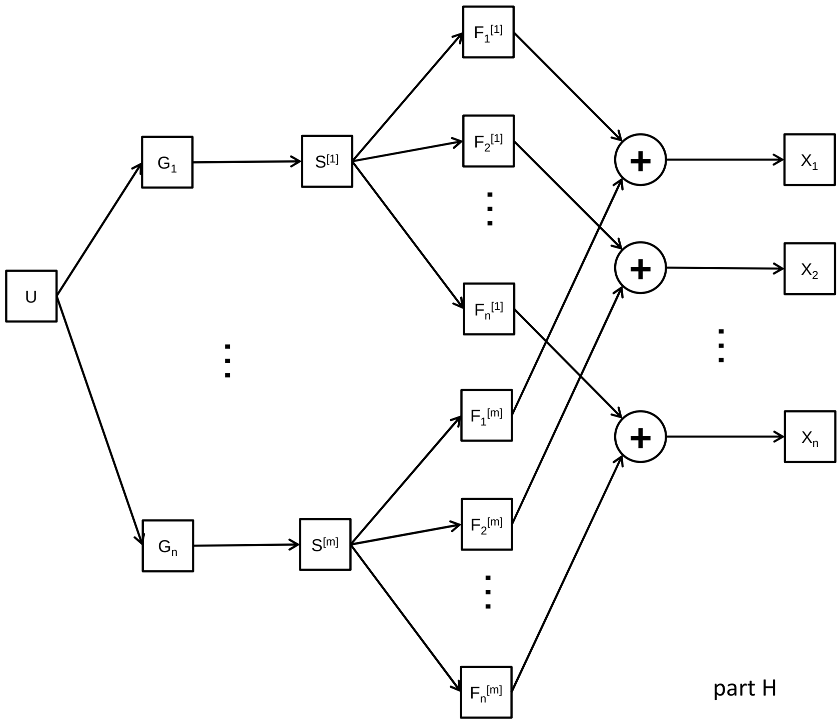


parts E-G

F. Consider the cross-spectra $P_{X_i X_j}(\omega)$ as a matrix $Z_m(\omega)$. What is the maximum number of nonzero eigenvalues that it can have? (i.e., what is its maximum rank?)

G. Is $Z_m(\omega)$ self-adjoint?

H. Now assume that the m noise sources $s^{[k]}(t)$ in E all arise from a single underlying signal $u(t)$, with $s^{[k]}(t)$ the result of filtering $u(t)$ by a linear filter G_k with transfer function $\tilde{G}_k(\omega)$. What is the maximum rank of the cross-spectral matrix $P_{X_i X_j}(\omega)$? (No need to derive this matrix, just justify the answer).



part H

Here, each output signal $x_i(t)$ is the result of filtering the same underlying noise $u(t)$ by a linear filter, albeit one that is more complicated than the one considered in A: its transfer function is $\sum_{k=1}^{m} \tilde{G}_k(\omega)\tilde{F}_i^{[k]}(\omega)$. So, as in A, the maximum rank of the cross-spectral matrix is 1.

5. Some properties of PCA

We start with the standard setup and formal solution for principal components analysis of a $n \times k$ matrix Y. For definiteness, think of Y as an array of k snapshots of images, each of which has n pixels. As in the notes, the principal-components decomposition is given by $Y = X\Lambda^{1/2}Z$ where X is $n \times p$ with orthonormal columns, Z is $p \times k$ with orthonormal rows, and Λ is a diagonal matrix whose entries, all non-negative, are the eigenvalues of YY^*.

As in the notes, one can either seek X as the first p (column) eigenvectors of the $n \times n$ matrix YY^* and then find $Z = \Lambda^{-1/2}X^*Y$, or seek Z as the first p (row) eigenvectors of the $k \times k$ matrix Y^*Y, and then find $X = YZ^*\Lambda^{-1/2}$.

Determine the effects of the following transformations on the principal-components decomposition, or, alternatively, whether the effects cannot be readily predicted; justify your answer.

A. Permuting the order of the snapshots
B. Permuting the order of the pixels
C. At each pixel, subtracting its mean throughout the dataset (i.e., subtracting the "average frame")
D. Replacing each pair of frames by their sum and their difference (assume an even number of frames)

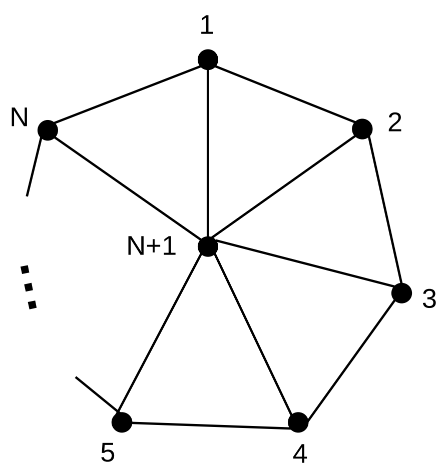
6. Group representations and graph Laplacians

Setup: a graph (bidirectional, unweighted) with an incidence matrix A, and a permutation that acts on the points of the graph, leaving the graph invariant, which we represent as a matrix P.

A. Show that P commutes with the graph Laplacian L .

B. Now let G be the group of all permutations that leave the graph invariant. There is a linear representation M of G in the space of functions on the graph: the permutation P corresponds to the linear transformation $\vec{x} \to P\vec{x}$, where $\vec{x}$ is a column vector of values assigned to each node. Say that this representation has an irreducible component M of dimension 1, and that $\vec{v}$ is a vector in this one-dimensional space. Show that $\vec{v}$ is an eigenvector of the graph Laplacian L.

C. Consider the the "wagonwheel" graph with $N+1$ nodes (one in the center, N on the rim), $N \geq 3$. . Write its graph Laplacian.



D. The cyclic group on N elements, considered to act on the N rim nodes, leaves the graph invariant. Recall that the cyclic group has irreducible representations M_r, in which a rotation by $2\pi k / N$ acts like multiplication by $e^{2\pi ikr/N}$, for each $r = 0,1,...,N-1$. For each nonzero r, find a function $\vec{v}^{[r]}$ on the graph for which rotation by $2\pi k / N$ acts like multiplication by $e^{2\pi ikr/N}$. Show that, up to scalar multiples, there is only one such function.

E. Find the eigenvalue of L corresponding to $\vec{v}^{[r]}$.

F. In D and E, we found $N-1$ eigenvectors and eigenvalues, one for each of the $N-1$ nontrivial representations of the cyclic group. So there are two more eigenvectors and eigenvalues to find, and they both must correspond to the trivial representation. Find the functions on the graph for which the permutation group acts trivially (i.e, leaves the functions unchanged).

G. Find two eigenvectors with distinct eigenvalues among the vectors identified in F.