Groups, Fields, and Vector Spaces

Homework #1 (2020-2021), Answers

Q1: Group or not a group?

Which are groups? If a group, is it commutative? Finite or infinite? If finite, how many elements? If infinite, is it discrete or continuous? If not a group, why not?

A. Complex numbers, under addition
Group, commutative, infinite, continuous.

B. Complex numbers, under multiplication
Not a group, inverse of 0 is does not exist.

C. The rotations of a regular pentagon into itself, under composition
Group, commutative, finite with 5 elements.

D. The rotations and mirror-reflections of a regular pentagon into itself, under composition
Group, non-commutative (since a mirror-reflection followed by a rotation does not yield the same result as a rotation followed by a mirror-reflection), 10 elements – the 5 rotations and the 5 mirrorings along planes that pass through one vertex and the midpoint of the opposite side.

E. The mirror-reflections of a regular pentagon into itself, under composition
Not a group, does not contain the identity, and is not closed under the group operation.

F. The integers $\{0,1,2,...,N-1\}$ *under addition* $\bmod N$ *, i.e.,* $a \circ b = c$ *if* $a+b-c$ *is a multiple of* N *.*
Group – and for $N=5$, abstractly the same group as Q1D.

G. The set of all translations of 3-space, under composition
Group, commutative, infinite, continuous.

H. The set of all rotations of 3-space, under composition
Group, non-commutative, infinite, continuous.

I. The set of all $N \times N$ *matrices with integer entries, under matrix addition*
Group, commutative, infinite, discrete.

J. The set of all $N \times N$ *matrices with integer entries, under matrix multiplication*

Not a group, does not contain inverses. The inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$\frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, so, unless $ad-bc=\pm 1$, inverses will not exist.

K. The set of all 2×2 matrices with integer entries and determinant 1, under matrix multiplication

Group, non-commutative, infinite, discrete. (See Q1J regarding inverses)

Q2. The "center"

The center of a group G , $Z(G)$, is defined as the set of elements c of G that commute with every element of G i..e, for which $g\circ c=c\circ g$.

A. Show that the center of a group is a group.

First, we need to show that the center is closed under the group operation, i.e., that if $c_1\in Z(G)$ and $c_2\in Z(G)$, then $c_1\circ c_2\in Z(G)$. To show this:

$g\circ(c_1\circ c_2)=(g\circ c_1)\circ c_2=(c_1\circ g)\circ c_2=c_1\circ(g\circ c_2)=c_1\circ(c_2\circ g)=(c_1\circ c_2)\circ g$, where each step follows either from associativity or from the assumption that $c_i\in Z(G)$. Next we need to show that the group axioms G1-G3 (notes pages 2-3) hold.

G1: Associativity. This holds in $Z(G)$ since it holds in G.

G2: Identity. We need to show that the identity element e for G is in the center. This holds because the group axioms for G imply that

$g\circ e=g=e\circ g$

G3: Inverses. We need to show that if $c\in G$ then $c^{-1}\in G$. This also follows from the group axioms for G. In complete detail (make sure you see which group axioms are used at each step:

$$
\begin{aligned}
& g\circ c=c\circ g \\
& \Rightarrow (g\circ c)\circ c^{-1}=(c\circ g)\circ c^{-1} \\
& \Rightarrow g\circ\left(c\circ c^{-1}\right)=c\circ\left(g\circ c^{-1}\right) \\
& \Rightarrow g=c\circ\left(g\circ c^{-1}\right) \\
& \Rightarrow c^{-1}\circ g=c^{-1}\circ\left(c\circ\left(g\circ c^{-1}\right)\right) \\
& \Rightarrow c^{-1}\circ g=\left(c^{-1}\circ c\right)\circ\left(g\circ c^{-1}\right) \\
& \Rightarrow c^{-1}\circ g=g\circ c^{-1}
\end{aligned}
$$

B. For each of the groups in Q1, find the center.

First, we note that if G is commutative, then $Z(G)=G$ so we only need to consider the non-commutative groups.

Q1D: First, note that (in general) $g \circ c = c \circ g$ is equivalent to $g \circ c \circ g^{-1} = c$. Note that a rotation followed by a reflection, followed by the inverse rotation, is a reflection across the rotated axis. So the only way that a rotation and a reflection will commute is if the rotation is through an angle of 0 or π. For the rotations of a regular N-gon, the rotations are $2\pi k / N$. So for N odd, only happens for $k = 0$, the trivial rotation. So $Z = \{e\}$. (For rotations and reflections of a regular N-gon where N is even, the center also contains the rotation by π.)

Q1H: A rotation g, followed by second rotation c, followed by the inverse rotation g^{-1}, is the same as a rotating by c after its axis that has been rotated by g. So unless c is trivial, there will be rotations that it does not commute with. So So $Z = \{e\}$.

Q1K. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be in the center, it must commute with all 2×2 matrices of integers with unit determinant. In particular,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} -b & a \\ -d & c \end{pmatrix} = \begin{pmatrix} c & d \\ -a & b \end{pmatrix}$$
, so $d = a$ and $c = -b$.

But also,

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 2a+b & a+b \\ a-2b & a-b \end{pmatrix} = \begin{pmatrix} 2a-b & a+2b \\ a-b & a+b \end{pmatrix}$$
, so $b = 0$.

So the only possibilities for the center are of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

Since the determinant is assumed to be 1, this means that $a = \pm 1$ and that

$$Z(G) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$