Groups, Fields, and Vector Spaces

Homework #3 (2020-2021), Answers

Q1: Homomorphism, or not?

In Q1A-C, V is the vector space of infinitely-differentiable functions on the reals, and f is a typical element. Which are the following are homomorphisms?

1A: $\varphi(f)=g$, *where* $g(x)=xf(x)$

Yes: We need to show $\varphi(\alpha_1 f_1+\alpha_2 f_2)=\alpha_1\varphi(f_1)+\alpha_2\varphi(f_2)$.

$$
\begin{aligned}
&\left(\varphi(\alpha_1 f_1+\alpha_2 f_2)\right)(x)=x\left((\alpha_1 f_1+\alpha_2 f_2)(x)\right)\\
&=x\left(\alpha_1 f_1(x)+\alpha_2 f_2(x)\right)\\
&=\alpha_1 x f_1(x)+\alpha_2 x f_2(x)\\
&=\alpha_1\left(\left(\varphi(f_1)\right)(x)\right)+\alpha_2\left(\left(\varphi(f_2)\right)(x)\right)
\end{aligned}
$$

(definition of φ, addition and scalar multiplication in *V*, definition of φ).
Since this holds for all $x\in\mathbb{R}$, then $\varphi(\alpha_1 f_1+\alpha_2 f_2)=\alpha_1\varphi(f_1)+\alpha_2\varphi(f_2)$.

1B. $\varphi(f)=g$, *where* $g(x)=f(x)+a$ *for some nonzero* a.

No. If we try to show $\varphi(\alpha_1 f_1+\alpha_2 f_2)=\alpha_1\varphi(f_1)+\alpha_2\varphi(f_2)$.

$\left(\varphi(\alpha_1 f_1+\alpha_2 f_2)\right)(x)=(\alpha_1 f_1+\alpha_2 f_2)(x)+a=\alpha_1 f_1(x)+\alpha_2 f_2(x)+a$, but

$$
\begin{aligned}
&\left(\alpha_1\varphi(f_1)+\alpha_2\varphi(f_2)\right)(x)=\alpha_1\left(\varphi(f_1)(x)\right)+\alpha_2\left(\varphi(f_2)(x)\right)\\
&=\alpha_1\left(f_1(x)+a\right)+\alpha_2\left(f_2(x)+a\right)\\
&=\alpha_1 f_1(x)+\alpha_2 f_2(x)+(\alpha_1+\alpha_2)a
\end{aligned}
.
$$

1C. $\varphi(f)=g$, *where* $g(x)=af(x)$ *for some nonzero* a.
Yes: We need to show $\varphi(\alpha_1 f_1+\alpha_2 f_2)=\alpha_1\varphi(f_1)+\alpha_2\varphi(f_2)$.

$$
\begin{aligned}
&\left(\varphi(\alpha_1 f_1+\alpha_2 f_2)\right)(x)=(\alpha_1 f_1+\alpha_2 f_2)(ax)\\
&=\left(\alpha_1 f_1(ax)+\alpha_2 f_2(ax)\right)\\
&=\alpha_1\left(\left(\varphi(f_1)\right)(x)\right)+\alpha_2\left(\left(\varphi(f_2)\right)(x)\right)
\end{aligned}
$$

(definition of φ, addition and scalar multiplication in *V*, definition of φ).
Since this holds for all $x\in\mathbb{R}$, then $\varphi(\alpha_1 f_1+\alpha_2 f_2)=\alpha_1\varphi(f_1)+\alpha_2\varphi(f_2)$.

1D. Here, the vector space V is the space of functions from a finite set $S=\{s_1,...,s_N\}$ to a field k, τ is a mapping from S to itself. For a vector $f \in V$, we define $\varphi(f)$ as the function on S given by $(\varphi(f))(s)=f(\tau(s))$. In words, φ acts on functions by relabeling their inputs according to τ. Is φ a homomorphism? Is it an isomorphism?

It is always a homomorphism.

$$\begin{aligned}&(\varphi(\alpha_1 f_1+\alpha_2 f_2))(s)=(\alpha_1 f_1+\alpha_2 f_2)(\tau(s))\\&=\alpha_1 f_1(\tau(s))+\alpha_2 f_2(\tau(s))\\&=\alpha_1((\varphi(f_1))(s))+\alpha_2((\varphi(f_2))(s))\end{aligned}.$$

(definition of φ, addition and scalar multiplication in V, definition of φ).

Since this holds for all $s\in S$, $\varphi(\alpha_1 f_1+\alpha_2 f_2)=\alpha_1\varphi(f_1)+\alpha_2\varphi(f_2)$. Note that this abstracts and generalizes 1C.

In order for φ to be an isomorphism, it has to be map onto all of V and be invertible. If, for example, τ mapped s_j and s_k onto the same s_0, then $(\varphi(f))(s_j)=f(\tau(s_j))=f(s_0)$, which would have to be the same as $(\varphi(f))(s_k)=f(\tau(s_k))=f(s_0)$. So the range of φ would not contain any elements whose values differed on s_j and s_k. Conversely, if τ mapped distinct elements of S onto distinct elements (i.e., is a permutation), then we can invert φ, by inverting the permutation: $(\varphi^{-1}(f))(s)=f(\tau^{-1}(s))$. So φ is an isomorphism if, and only if, τ is a permutation.

Q2: The transformation in $Hom(V,V)$ associated with coordinate transformations in V is an isomorphism.
In the notes, we found that, given a vector space V and a change in coordinates A (i.e., $v=Av'$), then there is an associated mapping in $Hom(V,V)$, Ψ_A, defined by $\Psi_A(L)=A^{-1}LA$.
A. Show that Ψ_A is an isomorphism in $Hom(V,V)$: that $\Psi_A(\alpha L)=\alpha\Psi_A(L)$ for any scalar α, that $\Psi_A(L+M)=\Psi_A(L)+\Psi_A(M)$ for any L and M in $Hom(V,V)$, and that Ψ_A has an inverse.

Scaling: $\Psi_A(\alpha L)=A^{-1}(\alpha L)A=\alpha A^{-1}LA=\alpha\Psi_A(L)$ (definition of Ψ_A, linearity of A^{-1}, definition of Ψ_A)

Superposition: $\Psi_A(L+M)=A^{-1}(L+M)A=A^{-1}LA+A^{-1}MA=\Psi_A(L)+\Psi_A(M)$ (definition of Ψ_A, linearity of A^{-1}, definition of Ψ_A)

Inverses: We show that $(\Psi_A)^{-1}$, the inverse of Ψ_A, is $\Psi_{A^{-1}}$

$\Psi_{A^{-1}}(\Psi_A(L))=(A^{-1})^{-1}\Psi_A(L)A^{-1}=A\Psi_A(L)A^{-1}=A(A^{-1}LA)A^{-1}=(AA^{-1})L(AA^{-1})=L$ (definition of $\Psi_{A^{-1}}$, inverse of inverse is self, definition of Ψ_A, associative law, definition of inverse). Since this holds for any L, $\Psi_{A^{-1}}$ is the inverse of Ψ_A

B. Is the mapping from A to Ψ_A linear? That is, is $\Psi_{\alpha A+\beta B}=\alpha\Psi_A+\beta\Psi_B$ for all scalars α, β and all isomorphisms A, B of V?

No. For example, for a nonzero scalar α, $\Psi_{\alpha A}(L)=\left(\alpha A\right)^{-1}L(\alpha A)=\alpha^{-1}A^{-1}L\alpha A=(\alpha^{-1}\alpha)A^{-1}LA=\Psi_A(L)$, but linearity would require that $\Psi_{\alpha A}=\alpha\Psi_A$.