Homework #3 (2020-2021), Answers

Q1: Homomorphism, or not?

In Q1A-C, \( V \) is the vector space of infinitely-differentiable functions on the reals, and \( f \) is a typical element. Which are the following homomorphisms?

1A. \( \varphi(f) = g \), where \( g(x) = xf(x) \)

1B. \( \varphi(f) = g \), where \( g(x) = f(x) + a \) for some nonzero \( a \).

1C. \( \varphi(f) = g \), where \( g(x) = af(x) \) for some nonzero \( a \).

1D. Here, the vector space \( V \) is the space of functions from a finite set \( S = \{s_1, \ldots, s_N\} \) to a field \( k \), \( \tau \) is a mapping from \( S \) to itself. For a vector \( f \in V \), we define \( \varphi(f) \) as the function on \( S \) given by \( (\varphi(f))(s) = f(\tau(s)) \). In words, \( \varphi \) acts on functions by relabeling their inputs according to \( \tau \). Is \( \varphi \) a homomorphism? Is it an isomorphism?

Q2: The transformation in \( \text{Hom}(V, V) \) associated with coordinate transformations in \( V \) is an isomorphism.

In the notes, we found that, given a vector space \( V \) and a change in coordinates \( A \) (i.e., \( v = Av' \)), then there is an associated mapping in \( \text{Hom}(V, V) \), \( \Psi_A \), defined by \( \Psi_A(L) = A^{-1}LA \).

A. Show that \( \Psi_A \) is an isomorphism in \( \text{Hom}(V, V) \): that \( \Psi_A(\alpha L) = \alpha \Psi_A(L) \) for any scalar \( \alpha \), that \( \Psi_A(L + M) = \Psi_A(L) + \Psi_A(M) \) for any \( L \) and \( M \) in \( \text{Hom}(V, V) \), and that \( \Psi_A \) has an inverse.

B. Is the mapping from \( A \) to \( \Psi_A \) linear? That is, is \( \Psi_{\alpha A + \beta B} = \alpha \Psi_A + \beta \Psi_B \) for all scalars \( \alpha \), \( \beta \) and all isomorphisms \( A \), \( B \) of \( V \)?