Groups, Fields, and Vector Spaces

Homework #3 (2020-2021), Answers

Q1: Homomorphism, or not?

In Q1A-C, V is the vector space of infintiely-differentiable functions on the reals, and f is a typical element. Which are the following are homomorphisms?

1A: $\varphi(f)=g$, where $g(x)=xf(x)$

1B. $\varphi(f)=g$, where $g(x)=f(x)+a$ for some nonzero a.

1C. $\varphi(f)=g$, where $g(x)=af(x)$ for some nonzero a.

1D. Here, the vector space V is the space of functions from a finite set $S=\{s_1,...,s_N\}$ to a field k, τ is a mapping from S to itself. For a vector $f\in V$, we define $\varphi(f)$ as the function on S given by $(\varphi(f))(s)=f(\tau(s))$. In words, φ acts on functions by relabeling their inputs according to τ. Is φ a homomorphism? Is it an isomorphism?

Q2: The transformation in $Hom(V,V)$ associated with coordinate transformations in V is an isomorphism. In the notes, we found that, given a vector space V and a change in coordinates A (i.e., $v=Av'$), then there is an associated mapping in $Hom(V,V)$, Ψ_A, defined by $\Psi_A(L)=A^{-1}LA$.

A. Show that Ψ_A is an isomorphism in $Hom(V,V)$: that $\Psi_A(\alpha L)=\alpha\Psi_A(L)$ for any scalar α, that $\Psi_A(L+M)=\Psi_A(L)+\Psi_A(M)$ for any L and M in $Hom(V,V)$, and that Ψ_A has an inverse.

B. Is the mapping from A to Ψ_A linear? That is, is $\Psi_{\alpha A+\beta B}=\alpha\Psi_A+\beta\Psi_B$ for all scalars α, β and all isomorphisms A, B of V?