Groups, Fields, and Vector Spaces (and a bit from Linear Transformations)

Homework #4 (2020-2021), Questions

Q1: Tensor products in 2 dimensions
The general setup is taken from the section of the notes on the effect of coordinate changes on tensor products: two vector spaces V and W, a coordinate change in V corresponding to an invertible $m \times m$ matrix A (with $v = Av'$, and a coordinate change in W corresponding to an invertible $n \times n$ matrix B (or a with $w = Bw'$).

We considered a tensor product $q \in V \otimes W$ expressed in coordinates, as $q = \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} (v_i \otimes w_j)$. We then showed that in the new coordinate system, $q = \sum_{k=1}^{m} \sum_{m=1}^{n} q_{k,l}{}' (v_k \otimes w_l)$, where the coefficients $q'_{k,l}$ of $v'_k \otimes w'_l$ are given by $q'_{k,l} = \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} A_{i,k} B_{j,l}$. Here, we specialize this to the case $V = W$, $A = B$, and $m = n = 2$.

A. Write out $q'_{k,l} = \sum_{i=1}^{m} \sum_{j=1}^{n} q_{i,j} A_{i,k} B_{j,l}$ explicitly for this special case (without summation notation).

B. Show that if $q_{i,j} = q_{j,i}$, then $q'_{k,l} = q'_{l,k}$. Note that this means we found a subspace of $V \otimes V$ that is invariant under all coordinate changes. What is its dimension?

C. Show that if $q_{i,j} = -q_{j,i}$, then $q'_{k,l} = -q'_{l,k}$. Note that this means we found another subspace of $V \otimes V$ that is invariant under all coordinate changes. What is its dimension?

Q2: Determinants of some transformations from first principles
Setup: V is a vector space of dimension m, and A is a linear transformation in $Hom(V,V)$. Further assume that A has m linearly-independent eigenvectors w_j, with $Aw_j = \lambda_j w_j$.

A. Find $\det(A)$ from the its definition as $\frac{anti((Av)^{\otimes m})}{anti(v^{\otimes m})}$.

B. With the above setup, find $\det(A \otimes A)$, where $A \otimes A$ is the linear transformation of $V \otimes V$ defined by $(A \otimes A)(v \otimes v') = Av \otimes Av'$.