Graph-Theoretic Methods

Homework #1 (2020-2021), Answers

Q1: Graph Laplacians

Consider the following graph.

A. Write its graph Laplacian $L$.
From the definition $L = D - A$, the difference of the diagonal matrix of the degrees ($D$) and the adjacency matrix,

$$
L = \begin{pmatrix}
+3 & -1 & -1 & -1 & 0 & 0 \\
-1 & +3 & -1 & 0 & -1 & 0 \\
-1 & -1 & +3 & 0 & 0 & -1 \\
-1 & 0 & 0 & +3 & -1 & -1 \\
0 & -1 & 0 & -1 & +3 & -1 \\
0 & 0 & -1 & -1 & -1 & +3
\end{pmatrix}
$$

B. Based on the symmetry of the graph, write a permutation matrix $P \neq I$ that commutes with $L$, for which $P^3 = I$.
Since the connectivity of the graph is unchanged following the permutation $(A_1A_2A_3)(A_4A_5A_6)$, we can take

$$
P = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

C. Determine the eigenvalues and eigenvectors of $P$.
Since $P^3 = I$, an eigenvector $v$ with eigenvalue $\lambda$ must satisfy $v = lv = P^3v = \lambda^3v$, so $\lambda^3 = 1$, and $\lambda = 1$ or $\lambda = \omega$ or $\lambda = \omega^2$, where $\omega = e^{2\pi i/3}$. 

Graph-Theoretic Methods 1 of 4
Regarding eigenvectors: $P$ is block-diagonal, so it acts independently on two three-dimensional subspaces. If, for example, $Pv = \lambda v$ -- in coordinates, \[ P \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}, \] then $x_2 = \lambda x_1$, $x_3 = \lambda x_2$, $x_1 = \lambda x_3$, and $x_5 = \lambda x_4$, $x_6 = \lambda x_5$, $x_4 = \lambda x_6$, so $v = \begin{pmatrix} a \\ \lambda a \\ \lambda^2 a \\ b \\ \lambda b \\ \lambda^2 b \end{pmatrix}$, where $\lambda \in \{1, \omega, \omega^2\}$, and at least one of $a$ and $b$ are nonzero. Choosing $(a, b) = (1,0)$ and $(a, b) = (0,1)$ thus displays six eigenvectors of $P$, as consisting of three pairs (one for each $\lambda \in \{1, \omega, \omega^2\}$). Within each pair, linear combinations are also eigenvectors. That is, the vector space in which $P$ and $L$ act is decomposed into three two-dimensional subspaces, one for each of the three distinct eigenvalues of $P$. See also Question 2 from Linear Transformations and Group Representations Homework 1 (LTGR2021aHW).

**D. Using $PL = LP$, determine the eigenvectors and eigenvalues of $L$.**

The eigenvectors for $L$ must lie wholly within each of the three two-dimensional subspaces outlined above. (Other than an intuitive argument based on symmetry), the formal reason for this is the following. If $Pv = \lambda v$, then $P(Lv) = L(Pv) = \lambda (Lv)$. So if $v$ is in the eigenspace corresponding to $\lambda$, then so is $Lv$. So we can see how $L$ acts on the eigenspace corresponding to $\lambda$ (where $\lambda \in \{1, \omega, \omega^2\}$) by choosing a convenient basis, \[ u_1(\lambda) = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } u_2(\lambda) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \lambda \\ \lambda^2 \\ \lambda^2 \end{pmatrix}, \] and a generic element of the eigenspace, $au_1 + bu_2$.

\[
L(au_1 + bu_2) = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 3 & \lambda^2 \end{pmatrix} \begin{pmatrix} a \\ \lambda a \\ \lambda^2 a \\ b \\ \lambda b \\ \lambda^2 b \end{pmatrix} = \begin{pmatrix} (3-\lambda-\lambda^2)a-b \\ (-1+3\lambda-\lambda^2)a-\lambda b \\ (-1-\lambda-3\lambda^2)a-\lambda^2 b \\ -a+(3-\lambda-\lambda^2)b \\ -a\lambda+(-1+3\lambda-\lambda^2)b \\ -a^2\lambda+(-1-\lambda+3\lambda^2)b \end{pmatrix} = \\
(3-\lambda-\lambda^2)a-b(u_1 + (-a+(3-\lambda-\lambda^2)b)u_2)
\]

Note that $\lambda^3 = 1$ was crucial for this, and this holds for $\lambda \in \{1, \omega, \omega^2\}$.
With \((a,b) = (1,0)\), \(Lu = (3 - \lambda - \lambda^2)u_1 - u_2\). With \((a,b) = (0,1)\), \(Lu = -u_1 + (3 - \lambda - \lambda^2)u_2\). So, in the eigenspace corresponding to \(\lambda\), \(L\) acts like the transformation

\[
L\lambda = \begin{pmatrix} 3 - \lambda - \lambda^2 & -1 \\ -1 & 3 - \lambda - \lambda^2 \end{pmatrix}.
\]

We therefore need to find the eigenvalues and eigenvectors of the above transformation, and can reconstitute them in the full 6-dimensional space from the definitions of \(u_1\) and \(u_2\).

Considering first \(\lambda = 1\) so \(L_i = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}\): The eigenvalues are \(\{2,0\}\), either by inspection or by solving the characteristic equation \(\det(\mu I - L_i) = \mu^2 - 2\mu = 0\) \(\mu = 2\) corresponds to the eigenvector \(u_1 - u_2 = \begin{pmatrix} +1 \\ +1 \\ +1 \\ -1 \\ -1 \\ -1 \end{pmatrix}\), \(\mu = 0\) corresponds to the eigenvector \(u_1 + u_2 = \begin{pmatrix} +1 \\ +1 \\ +1 \\ +1 \\ +1 \\ +1 \end{pmatrix}\), the trivial eigenvector of a connected graph.

\(\lambda = \omega\) and \(\lambda = \omega^2\) can be considered together, as \(\omega\) and \(\omega^2\) are complex-conjugates.

\[
L_\omega = \begin{pmatrix} 3 - \omega - \omega^2 & -1 \\ -1 & 3 - \omega - \omega^2 \end{pmatrix} = \begin{pmatrix} +4 & -1 \\ -1 & +4 \end{pmatrix},
\]

since \(1 + \omega + \omega^2 = 0\). Solving the characteristic equation \(\det(\mu I - L_\omega) = \mu^2 - 8\mu + 15 = 0\) yields eigenvalues of 3 and 5, where 5 corresponds to the eigenvector \(u_1 - u_2 = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ -1 \\ -\omega \\ -\omega^2 \end{pmatrix}\) and 3 corresponds to the eigenvector \(u_1 + u_2 = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ 1 \\ \omega \\ \omega^2 \end{pmatrix}\).

So in sum, we have six eigenvalues, two pairs of which are identical, with the following eigenvectors:

\[
\mu = 0: \begin{pmatrix} +1 \\ +1 \\ +1 \\ +1 \\ +1 \\ +1 \end{pmatrix}^T \text{ in the } \lambda \text{-eigenspace of } P
\]

\[
\mu = 2: \begin{pmatrix} +1 \\ +1 \\ +1 \\ -1 \\ -1 \\ -1 \end{pmatrix}^T 
\]

\[
\mu = 3: \begin{pmatrix} +1 \\ +\omega \\ +\omega^2 \\ +1 \\ +\omega \\ +\omega^2 \end{pmatrix}^T \text{ and its complex conjugate}\n\]

\[
\mu = 5: \begin{pmatrix} +1 \\ +\omega \\ +\omega^2 \\ -1 \\ -\omega \\ -\omega^2 \end{pmatrix}^T \text{ and its complex conjugate}\n\]
If we want real eigenvectors, we can add and subtract the complex eigenvectors and their complex-conjugates, yielding (for example) an eigenvector \((+2 \ -1 \ -1 \ +2 \ -1 \ -1)^T\) of eigenvalue 3.

Finally, note also that there is another permutation matrix, \(Q\), of period two, that commutes with \(L\). Why did we use \(P\) instead of \(Q\)?