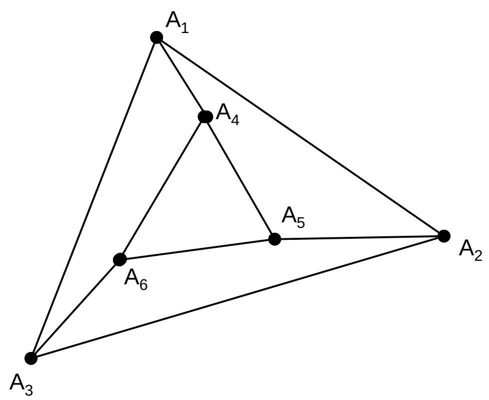
Graph-Theoretic Methods

Homework #1 (2020-2021), Answers

Q1: Graph Laplacians

Consider the following graph.



A. Write its graph Laplacian L.

From the definition $L = D - A$, the difference of the diagonal matrix of the degrees (D) and the adjacency matrix,

$$L = \begin{pmatrix} +3 & -1 & -1 & -1 & 0 & 0 \\ -1 & +3 & -1 & 0 & -1 & 0 \\ -1 & -1 & +3 & 0 & 0 & -1 \\ -1 & 0 & 0 & +3 & -1 & -1 \\ 0 & -1 & 0 & -1 & +3 & -1 \\ 0 & 0 & -1 & -1 & -1 & +3 \end{pmatrix}$$

B. Based on the symmetry of the graph, write a permutation matrix $P \neq I$ *that commutes with* L*, for which* $P^3 = I$.

Since the connectivity of the graph is unchanged following the permutation $(A_1A_2A_3)(A_4A_5A_6)$, we can take

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

C. Determine the eigenvalues and eigenvectors of P.

Since $P^3 = I$, an eigenvector v with eigenvalue λ must satisfy $v = Iv = P^3v = \lambda^3 v$, so $\lambda^3 = 1$, and $\lambda = 1$ or $\lambda = \omega$ or $\lambda = \omega^2$, where $\omega = e^{2\pi i/3}$.

Regarding eigenvectors: P is block-diagonal, so it acts independently on two three-dimensional subspaces. If, for example, $Pv = \lambda v$ -- in coordinates, $P\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$ -- , then $x_2 = \lambda x_1$, $x_3 = \lambda x_2$, $x_1 = \lambda x_3$, and $x_5 = \lambda x_4$, $x_6 = \lambda x_5$, $x_4 = \lambda x_6$, so $v = \begin{pmatrix} a \\ \lambda a \\ \lambda^2 a \\ b \\ \lambda b \\ \lambda^2 b \end{pmatrix}$, where $\lambda \in \{1, \omega, \omega^2\}$, and at least one of a and b are nonzero. Choosing $(a,b) = (1,0)$ and $(a,b) = (0,1)$ thus displays six eigenvectors of P, as consisting of three pairs (one for each $\lambda \in \{1, \omega, \omega^2\}$). Within each pair, linear combinations are also eigenvectors. That is, the vector space in which P and L act is decomposed into three two-dimensional subspaces, one for each of the three distinct eigenvalues of P. See also Question 2 from Linear Transformations and Group Representations Homework 1 (LTGR2021aHW).

D. Using $PL = LP$, determine the eigenvectors and eigenvalues of L.

The eigenvectors for L must lie wholly within each of the three two-dimensional subspaces outlined above. (Other than an intuitive argument based on symmetry), the formal reason for this is the following. If $Pv = \lambda v$, then $P(Lv) = L(Pv) = \lambda(Lv)$. So if v is in the eigenspace corresponding to λ, then so is Lv. So we can see how L acts on the eigenspace corresponding to λ (where $\lambda \in \{1, \omega, \omega^2\}$) by choosing a convenient basis, $u_1(\lambda) = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $u_2(\lambda) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \lambda \\ \lambda^2 \end{pmatrix}$, and a generic element of the eigenspace, $au_1 + bu_2$.

$$L(au_1 + bu_2) = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} a \\ \lambda a \\ \lambda^2 a \\ b \\ \lambda b \\ \lambda^2 b \end{pmatrix} = \begin{pmatrix} (3-\lambda-\lambda^2)a - b \\ (-1+3\lambda-\lambda^2)a - \lambda b \\ (-1-\lambda-3\lambda^2)a - \lambda^2 b \\ -a + (3-\lambda-\lambda^2)b \\ -a\lambda + (-1+3\lambda-\lambda^2)b \\ -a^2\lambda + (-1-\lambda+3\lambda^2)b \end{pmatrix} =$$

$$= \left((3-\lambda-\lambda^2)a - b\right)u_1 + \left(-a + (3-\lambda-\lambda^2)b\right)u_2$$

Note that $\lambda^3 = 1$ was crucial for this, and this holds for $\lambda \in \{1, \omega, \omega^2\}$.

With $(a,b)=(1,0)$, $Lu_1=(3-\lambda-\lambda^2)u_1-u_2$. With $(a,b)=(0,1)$, $Lu_2=-u_1+(3-\lambda-\lambda^2)u_2$. So, in the eigenspace corresponding to λ, L acts like the transformation

$$L_\lambda=\begin{pmatrix}3-\lambda-\lambda^2 & -1\\ -1 & 3-\lambda-\lambda^2\end{pmatrix}.$$

We therefore need to find the eigenvalues and eigenvectors of the above transformation, and can reconstitute them in the full 6-dimensional space from the definitions of u_1 and u_2.

Considering first $\lambda=1$ so $L_1=\begin{pmatrix}+1 & -1\\ -1 & +1\end{pmatrix}$: The eigenvalues are $\{2,0\}$, either by inspection or by solving the characteristic equation $\det(\mu I-L_1)=\det\begin{pmatrix}\mu-1 & 1\\ 1 & \mu-1\end{pmatrix}=\mu^2-2\mu=0$ $\mu=2$ corresponds to the eigenvector $u_1-u_2=\begin{pmatrix}+1\\+1\\+1\\-1\\-1\\-1\end{pmatrix}$, $\mu=0$ corresponds to the eigenvector $u_1+u_2=\begin{pmatrix}+1\\+1\\+1\\+1\\+1\\+1\end{pmatrix}$, the trivial eigenvector of a connected graph.

$\lambda=\omega$ and $\lambda=\omega^2$ can be considered together, as ω and ω^2 are complex-conjugates.
$L_\omega=\begin{pmatrix}3-\omega-\omega^2 & -1\\ -1 & 3-\omega-\omega^2\end{pmatrix}=\begin{pmatrix}+4 & -1\\ -1 & +4\end{pmatrix}$, since $1+\omega+\omega^2=0$. Solving the characteristic equation $\det(\mu I-L_\omega)=\det\begin{pmatrix}\mu-4 & 1\\ 1 & \mu-4\end{pmatrix}=\mu^2-8\mu+15=0$ yields eigenvalues of 3 and 5, where 5 corresponds to the eigenvector $u_1-u_2=\begin{pmatrix}1\\ \omega\\ \omega^2\\ -1\\ -\omega\\ -\omega^2\end{pmatrix}$ and 3 corresponds to the eigenvector $u_1+u_2=\begin{pmatrix}1\\ \omega\\ \omega^2\\ 1\\ \omega\\ \omega^2\end{pmatrix}$.

So in sum, we have six eigenvalues, two pairs of which are identical, with the following eigenvectors:

$$\left.\begin{array}{l}\mu=0:\begin{pmatrix}+1 & +1 & +1 & +1 & +1 & +1\end{pmatrix}^T\\ \mu=2:\begin{pmatrix}+1 & +1 & +1 & -1 & -1 & -1\end{pmatrix}^T\end{array}\right\}in\,the\,1-eigenspace\,of\;P$$

$$\left.\begin{array}{l}\mu=3:\begin{pmatrix}+1 & +\omega & +\omega^2 & +1 & +\omega & +\omega^2\end{pmatrix}^T\ and\ its\ complex-conjugate\\ \mu=5:\begin{pmatrix}+1 & +\omega & +\omega^2 & -1 & -\omega^2 & -\omega^2\end{pmatrix}^T\ and\ its\ complex-conjugate\end{array}\right\}in\ the\ \omega-and\ \omega^2-eigenspaces\ of\ P.$$

If we want real eigenvectors, we can add and subtract the complex eigenvectors and their complex-conjugates, yielding (for example) an eigenvector $\begin{pmatrix}+2 & -1 & -1 & +2 & -1 & -1\end{pmatrix}^T$ of eigenvalue 3.

Finally, note also that there is another permutation matrix, Q, of period two, that commutes with L. Why did we use P instead of Q?