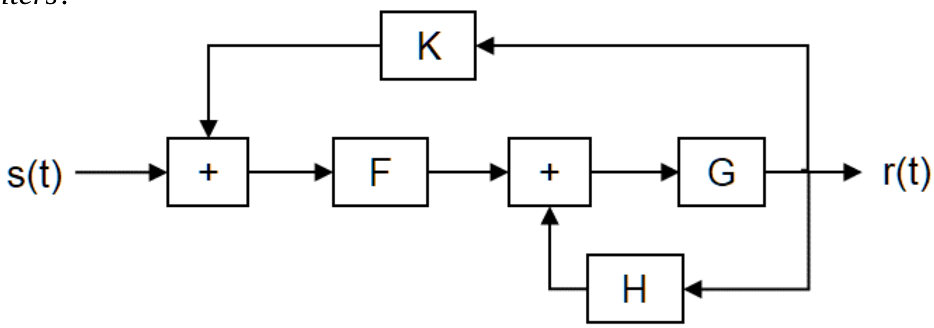
Linear Systems and Black Boxes

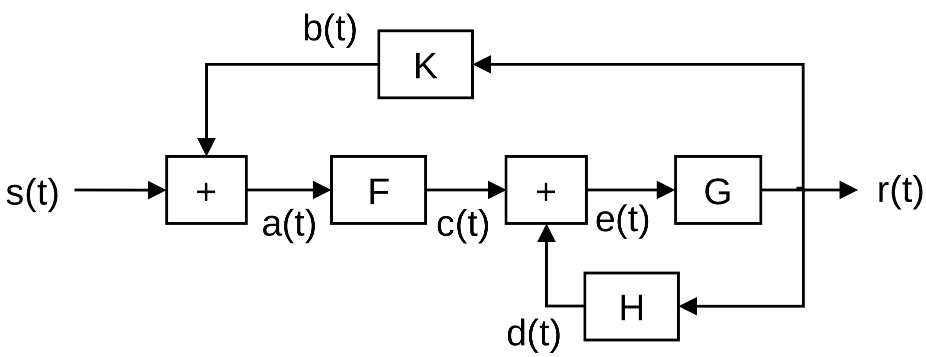
Homework #1 (2020-2021), Answers

Q1: Transfer function of a more complex composite system

Consider the following composite system, with linear filters F, G, H, and K, input $s(t)$ and output $r(t)$. What is the transfer function that relates $\tilde{r}(\omega)$ to $\tilde{s}(\omega)$, in terms of the transfer functions of the component filters?



Label the intermediate signals as follows:



We now chase signals around until we have sufficient dependencies:

First summing box: $\tilde{a}(\omega)=\tilde{s}(\omega)+\tilde{b}(\omega)$

$F:\ \tilde{c}(\omega)=\tilde{F}(\omega)\tilde{a}(\omega)=\tilde{F}(\omega)\left(\tilde{s}(\omega)+\tilde{b}(\omega)\right)$

Second summing box: $\tilde{e}(\omega)=\tilde{c}(\omega)+\tilde{d}(\omega)=\tilde{F}(\omega)\left(\tilde{s}(\omega)+\tilde{b}(\omega)\right)+\tilde{d}(\omega)$

$G:\ \tilde{r}(\omega)=\tilde{G}(\omega)\tilde{e}(\omega)=\tilde{F}(\omega)\tilde{G}(\omega)\left(\tilde{s}(\omega)+\tilde{b}(\omega)\right)+\tilde{G}(\omega)\tilde{d}(\omega)$

$H:\tilde{d}(\omega)=\tilde{H}(\omega)\tilde{r}(\omega)=\tilde{F}(\omega)\tilde{G}(\omega)\tilde{H}(\omega)\left(\tilde{s}(\omega)+\tilde{b}(\omega)\right)+\tilde{G}(\omega)\tilde{H}(\omega)\tilde{d}(\omega)$.

The last equation allows us to solve for $\tilde{d}(\omega)$:

$\tilde{d}(\omega)\left(1-\tilde{G}(\omega)\tilde{H}(\omega)\right)=\tilde{F}(\omega)\tilde{G}(\omega)\tilde{H}(\omega)\left(\tilde{s}(\omega)+\tilde{b}(\omega)\right)$, so $\tilde{d}(\omega)=\dfrac{\tilde{F}(\omega)\tilde{G}(\omega)\tilde{H}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}\left(\tilde{s}(\omega)+\tilde{b}(\omega)\right)$.

$K:\tilde{b}(\omega)=\tilde{K}(\omega)\tilde{r}(\omega)$.

Now substituting the equations for $\tilde{b}$ and $\tilde{d}$ in the equation for G:

$$\tilde{r}(\omega)=\tilde{F}(\omega)\tilde{G}(\omega)\left(\tilde{s}(\omega)+\tilde{b}(\omega)\right)+\tilde{G}(\omega)\tilde{d}(\omega)$$

$$=\tilde{F}(\omega)\tilde{G}(\omega)\tilde{s}(\omega)+\tilde{F}(\omega)\tilde{G}(\omega)\tilde{K}(\omega)\tilde{r}(\omega)+\tilde{G}(\omega)\frac{\tilde{F}(\omega)\tilde{G}(\omega)\tilde{H}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}\left(\tilde{s}(\omega)+\tilde{K}(\omega)\tilde{r}(\omega)\right)$$

$$=\frac{\tilde{F}(\omega)\tilde{G}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}\tilde{s}(\omega)+\tilde{F}(\omega)\tilde{G}(\omega)\tilde{K}(\omega)\tilde{r}(\omega)+\tilde{G}(\omega)\frac{\tilde{F}(\omega)\tilde{G}(\omega)\tilde{H}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}\tilde{K}(\omega)\tilde{r}(\omega)$$

$$=\frac{\tilde{F}(\omega)\tilde{G}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}\tilde{s}(\omega)+\frac{\tilde{F}(\omega)\tilde{G}(\omega)\tilde{K}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}\tilde{r}(\omega)$$

So

$$\tilde{r}(\omega)=\tilde{F}(\omega)\tilde{G}(\omega)\left(\tilde{s}(\omega)+\tilde{b}(\omega)\right)+\tilde{G}(\omega)\tilde{d}(\omega)$$

$$=\tilde{F}(\omega)\tilde{G}(\omega)\tilde{s}(\omega)+\tilde{F}(\omega)\tilde{G}(\omega)\tilde{K}(\omega)\tilde{r}(\omega)+\tilde{G}(\omega)\frac{\tilde{F}(\omega)\tilde{G}(\omega)\tilde{H}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}\left(\tilde{s}(\omega)+\tilde{K}(\omega)\tilde{r}(\omega)\right)$$

$$=\frac{\tilde{F}(\omega)\tilde{G}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}\tilde{s}(\omega)+\tilde{F}(\omega)\tilde{G}(\omega)\tilde{K}(\omega)\tilde{r}(\omega)+\tilde{G}(\omega)\frac{\tilde{F}(\omega)\tilde{G}(\omega)\tilde{H}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}\tilde{K}(\omega)\tilde{r}(\omega)$$

$$=\frac{\tilde{F}(\omega)\tilde{G}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}\tilde{s}(\omega)+\frac{\tilde{F}(\omega)\tilde{G}(\omega)\tilde{K}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}\tilde{r}(\omega)$$

and

$$\tilde{r}(\omega)=\frac{\dfrac{\tilde{F}(\omega)\tilde{G}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}}{1-\dfrac{\tilde{F}(\omega)\tilde{G}(\omega)\tilde{K}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)}}\tilde{s}(\omega)=\frac{\tilde{F}(\omega)\tilde{G}(\omega)}{1-\tilde{G}(\omega)\tilde{H}(\omega)-\tilde{F}(\omega)\tilde{G}(\omega)\tilde{K}(\omega)}\tilde{s}(\omega).$$

As a check, one can look at some special cases in which this reduces to a simple feedback system: $\tilde{H}(\omega)=0$, $\tilde{K}(\omega)=0$, or $\tilde{F}(\omega)=1$.