Q1: Biased random walks

Here we extend the analysis in the notes to a random walk with a directional bias. We allow the particle to move a step of size \( b \) to the right or left, but with unequal probability, so that the probability distribution at time \( t + \Delta T \) is related to the probability at time \( t \) by convolution with

\[
\hat{F}_{\Delta T}(x) = \int F_{\Delta T}(x') e^{-ix' \omega} dx' = \int F_{\Delta T}(x) e^{-ix \omega} dx = \int \frac{1}{2}((1-c)\delta(x-b) + (1+c)\delta(x+b)) e^{-ix \omega} dx
\]

where \( c = \frac{1}{2}(1-c) \) and \( p_+ = \frac{1}{2}(1+c) \).

A. Calculate

\[
\hat{F}_{\Delta T}(\omega) = \int F_{\Delta T}(x) e^{-ix \omega} dx
\]

\[
\hat{F}_{\Delta T}(\omega) = \int F_{\Delta T}(x) e^{-ix \omega} dx = \int \frac{1}{2}((1-c)\delta(x-b) + (1+c)\delta(x+b)) e^{-ix \omega} dx
\]

\[
= \frac{1}{2}((1-c)e^{-ib \omega} + (1+c)e^{ib \omega}) = \frac{1}{2}(e^{-ib \omega} + e^{ib \omega}) + \frac{1}{2}e^{-ib \omega} + \frac{1}{2}e^{ib \omega} = \cos(\omega b) + ic \sin(\omega b)
\]

where the last step used \( \cos u = \frac{e^{iu} + e^{-iu}}{2} \) and \( \sin u = \frac{e^{iu} - e^{-iu}}{2i} \).

B. Determine how the probability distribution evolves over time \( T \) by determining

\[
\frac{\hat{F}_{\Delta T}(\omega)}{\hat{F}_{\Delta T}(\omega)} \bigg|_{\Delta T \to 0} = \lim_{\Delta T \to 0} \left( \int F_{\Delta T}(x) e^{-ix \omega} dx \right)^{T/\Delta T}
\]

As \( \Delta T \to 0 \), both \( b \) and \( c \) are small. For small \( b \) and \( c \),

\[
\hat{F}_{\Delta T}(\omega) = \cos(\omega b) + ic \sin(\omega b)
\]

\[
\approx 1 - \frac{1}{2} \omega^2 b^2 + 2ic \omega = 1 - \frac{1}{2} \omega^2 A \Delta T + i\omega\sqrt{AC} \Delta T,
\]

with the omitted terms decreasing as \((\Delta T)^2\) or faster. So

\[
\lim_{\Delta T \to 0} \left( \hat{F}_{\Delta T}(\omega) \right)^{T/\Delta T} = \lim_{\Delta T \to 0} \left( 1 - \frac{1}{2} \omega^2 A - i\omega\sqrt{AC} \right)^{T/\Delta T} = \exp \left( \frac{1}{2} \omega^2 A T - i\omega T\sqrt{AC} \right).
\]

C. Given a distribution \( q_0(x) = \delta(x) \) at time 0, determine the distribution at time \( T \) via Fourier synthesis.

\[
\hat{q}_T(\omega) = \hat{F}_\omega(\omega) \hat{q}(0) = e^{-\omega^2 AT/2 - i\omega T\sqrt{AC}}.
\]

The Fourier synthesis for the unbiased walk (in the notes) will be helpful.

\[
q_T(x) = \frac{1}{2\pi} \int \hat{q}_T(\omega) e^{ix \omega} d\omega = \frac{1}{2\pi} \int e^{-\omega^2 AT/2 - i\omega T\sqrt{AC}} e^{ix \omega} d\omega = \frac{1}{2\pi} \int e^{-\omega^2 AT/2 + i\omega(x - T\sqrt{AC})} d\omega.
\]

Note that this is the same integral as in the text for the symmetric case,

\[
p_T(x) = \frac{1}{2\pi} \int \hat{p}_T(\omega) e^{ix \omega} d\omega = \frac{1}{2\pi} \int e^{-\omega^2 AT/2} e^{ix \omega} d\omega = \frac{1}{\sqrt{2\pi AT}} e^{-x^2/2AT},
\]

except that \( x \) is replaced by \( x - T\sqrt{AC} \). So \( q_T(x) = p_T(x - T\sqrt{AC}) = \frac{1}{\sqrt{2\pi AT}} e^{-(x-T\sqrt{AC})^2/2AT} \). That is, the bias adds a positional offset that increases linearly with time.
Q2: Another biased random walk

In this random walk, the step probabilities are equal, but the step sizes are not: So the probability distribution at time \( t + \Delta T \) is related to the probability at time \( t \) by convolution with \( F_{\Delta T}(x) = \frac{1}{2} (\delta(x - b_+) + \delta(x + b_-)) \), where \( b_- = b - s \), and \( b_+ = b + s \).

A. Calculate \( \hat{F}_{\Delta T}(\omega) = \int_{-\infty}^{\infty} F_{\Delta T}(x)e^{-ix\omega} dx \)

\[
\hat{F}_{\Delta T}(\omega) = \int_{-\infty}^{\infty} F_{\Delta T}(x)e^{-ix\omega} dx = \int_{-\infty}^{\infty} \frac{1}{2} (\delta(x - b_+) + \delta(x + b_-))e^{-ix\omega} dx
\]

\[
= \frac{1}{2} (e^{-i\omega b + i\omega s} + e^{i\omega b + i\omega s}) = \frac{1}{2} (e^{-i\omega b} + e^{i\omega b})e^{i\omega s}
\]

where the last step evaluated the integrand at \( x = b_- = b - s \) and \( x = -b_+ = -b - s \).

B. Determine how the probability distribution evolves over time \( T \) by determining \( \left( \hat{F}_{\Delta T}(\omega) \right)^{T/\Delta T} \) in the limit of \( \Delta T \to 0 \) with (as in the text) \( b^2 = A\Delta T \) but also \( s = S\Delta T \).

As \( \Delta T \to 0 \), both \( b \) and \( s \) are small. For small \( b \) and \( s \),

\[
\hat{F}_{\Delta T}(\omega) = \cos(\omega b)e^{i\omega s} \approx \left(1 - \frac{1}{2} \omega^2 b^2 \right) (1 + i\omega s) \approx 1 - \frac{1}{2} \omega^2 A\Delta T + i\omega S\Delta T,
\]

with the omitted terms decreasing as \((\Delta T)^2\) or faster. So

\[
\lim_{\Delta T \to 0} \left( \hat{F}_{\Delta T}(\omega) \right)^{T/\Delta T} = \lim_{\Delta T \to 0} \left( 1 - \frac{1}{2} \omega^2 A - i\omega S \right)^{T/\Delta T} = \exp \left( -\frac{1}{2} \omega^2 A T - i\omega S T \right)
\]

C. Can this behavior be distinguished from that of the biased random walk in Q1? Why or why not?

No, they cannot be distinguished as their evolution with time is identical: with \( S = \sqrt{AT} \) the expressions for \( \lim_{\Delta T \to 0} \left( \hat{F}_{\Delta T}(\omega) \right)^{T/\Delta T} \) are equal.