Linear Systems and Black Boxes

Homework #2 (2020-2021), Answers

Q1: Biased random walks

Here we extend the analysis in the notes to a random walk with a directional bias. We allow the particle to move a step of size b to the right or left, but with unequal probability, so that the probability distribution at time $t+\Delta T$ is related to the probability at time t by convolution with $F_{\Delta T}(x) = \left(p_{\_}\delta(x-b) + p_{+}\delta(x+b)\right)$, where $p_{-} = \frac{1}{2}(1-c)$ and $p_{+} = \frac{1}{2}(1+c)$.

A. Calculate $\hat{F}_{\Delta T}(\omega) = \int_{-\infty}^{\infty} F_{\Delta T}(x) e^{-i\omega x} dx$

$$\hat{F}_{\Delta T}(\omega) = \int_{-\infty}^{\infty} F_{\Delta T}(x) e^{-i\omega x} dx = \int_{-\infty}^{\infty} \frac{1}{2}\left((1-c)\delta(x-b) + (1+c\delta)(x+b)\right) e^{-i\omega x} dx$$
$$= \frac{1}{2}\left((1-c)e^{-i\omega b} + (1+c)e^{i\omega b}\right) = \frac{1}{2}\left(e^{-i\omega b} + e^{i\omega b}\right) + \frac{1}{2}c\left(-e^{-i\omega b} + e^{i\omega b}\right) = \cos(\omega b) + ic\sin(\omega b),$$

where the last step used $\cos u = \dfrac{e^{iu} + e^{-iu}}{2}$ and $\sin u = \dfrac{e^{iu} - e^{-iu}}{2i}$.

B. Determine how the probability distribution evolves over time T by determining $\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T}$ in the limit of $\Delta T \to 0$ with (as in the text) $b^2 = A\Delta T$ but also $c^2 = C\Delta T$.

As $\Delta T \to 0$, both b and c are small. For small b and c, $\hat{F}_{\Delta T}(\omega) = \cos(\omega b) + ic\sin(\omega b) \approx 1 - \frac{1}{2}\omega^2 b^2 + ibc\omega = 1 - \frac{1}{2}\omega^2 A\Delta T + i\omega\sqrt{AC}\Delta T$, with the omitted terms decreasing as $(\Delta T)^2$ or faster. So

$$\lim_{\Delta T\to 0}\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T} = \lim_{\Delta T\to 0}\left(1 - \left(\frac{1}{2}\omega^2 A - i\omega\sqrt{AC}\right)\Delta T\right)^{T/\Delta T} = \exp - \left(\frac{1}{2}\omega^2 AT - i\omega T\sqrt{AC}\right).$$

C. Given a distribution $q_0(x) = \delta(x)$ at time 0, determine the distribution at time T via Fourier synthesis.

$\hat{q}_T(\omega) = \hat{F}_T(\omega)\hat{q}(0) = e^{-\omega^2 AT/2 - i\omega T\sqrt{AC}}$. *The Fourier synthesis for the unbiased walk (in the notes) will be helpful.*

$$q_T(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{q}_T(\omega) e^{i\omega x} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\omega^2 AT/2 - i\omega T\sqrt{AC}} e^{i\omega x} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\omega^2 AT/2 + i\omega(x - T\sqrt{AC})} d\omega.$$

Note that this is the same integral as in the text for the symmetric case, $p_T(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{p}_T(\omega) e^{i\omega x} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\omega^2 AT/2} e^{i\omega x} d\omega = \frac{1}{\sqrt{2\pi AT}} e^{-x^2/2AT}$, except that x is replaced by $x - T\sqrt{AC}$. So $q_T(x) = p_T(x - T\sqrt{AC}) = \frac{1}{\sqrt{2\pi AT}} e^{-(x - T\sqrt{AC})^2/2AT}$. That is, the bias adds a positional offset that increases linearly with time.

Q2: Another biased random walk

In this random walk, the step probabilities are equal, but the step sizes are not: So the probability distribution at time $t+\Delta T$ *is related to the probability at time* t *by convolution with* $F_{\Delta T}(x)=\frac{1}{2}\left(\delta(x-b_-)+\delta(x+b_+)\right)$, *where* $b_-=b-s$ *,and* $b_+=b+s$.

A. Calculate $\hat{F}_{\Delta T}(\omega)=\int_{-\infty}^{\infty}F_{\Delta T}(x)e^{-i\omega x}dx$

$$\hat{F}_{\Delta T}(\omega)=\int_{-\infty}^{\infty}F_{\Delta T}(x)e^{-i\omega x}dx=\int_{-\infty}^{\infty}\frac{1}{2}\left(\delta(x-b_-)+\delta(x+b_+)\right)e^{-i\omega x}dx$$

$$=\frac{1}{2}\left(e^{-i\omega b+i\omega s}+e^{i\omega b+i\omega s}\right)=\frac{1}{2}\left(e^{-i\omega b}+e^{i\omega b}\right)e^{i\omega s}$$

where the last step evaluated the integrand at $x=b_-=b-s$ and $x=-b_+=-b-s$.

B. Determine how the probability distribution evolves over time T *by determining* $\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T}$ *in the limit of* $\Delta T\to 0$ *with (as in the text)* $b^2=A\Delta T$ *but also* $s=S\Delta T$.

As $\Delta T\to 0$, both b and s are small. For small b and s,
$\hat{F}_{\Delta T}(\omega)=\cos(\omega b)e^{i\omega s}\approx\left(1-\frac{1}{2}\omega^2 b^2\right)\left(1+i\omega s\right)\approx 1-\frac{1}{2}\omega^2 A\Delta T+i\omega S\Delta T$, with the omitted terms decreasing as $\left(\Delta T\right)^2$ or faster. So

$$\lim_{\Delta T\to 0}\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T}=\lim_{\Delta T\to 0}\left(1-\left(\frac{1}{2}\omega^2 A-i\omega S\right)\Delta T\right)^{T/\Delta T}=\exp-\left(\frac{1}{2}\omega^2 AT-i\omega ST\right)$$

.

C. Can this behavior be distinguished from that of the biased random walk in Q1? Why or why not?

No, they cannot be distinguished as their evolution with time is identical: with $S=\sqrt{AT}$ the expressions for $\lim_{\Delta T\to 0}\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T}$ are equal.