Linear Systems and Black Boxes

Homework #2 (2020-2021), Questions

Q1: Biased random walks

Here we extend the analysis in the notes to a random walk with a directional bias. We allow the particle to move a step of size $b$ to the right or left, but with unequal probability, so that the probability distribution at time $t + \Delta T$ is related to the probability at time $t$ by convolution with $F_{\Delta T}(x) = \left(p_- \delta(x - b) + p_+ \delta(x + b)\right)$, where $p_- = \frac{1}{2}(1-c)$ and $p_+ = \frac{1}{2}(1+c)$.

A. Calculate $\hat{F}_{\Delta T}(\omega) = \int_{-\infty}^{\infty} F_{\Delta T}(x) e^{-i\omega x} dx$

B. Determine how the probability distribution evolves over time $T$ by determining $\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T}$ in the limit of $\Delta T \to 0$ with (as in the text) $b^2 = A\Delta T$ but also $c^2 = C\Delta T$.

C. Given a distribution $q_0(x) = \delta(x)$ at time 0, determine the distribution at time $T$ via Fourier synthesis. $\hat{q}_T(\omega) = \hat{F}_T(\omega)\hat{q}(0) = e^{-\frac{\omega^2 A T^2}{2} - i\omega \Delta T \hat{F}}$. The Fourier synthesis for the unbiased walk (in the notes) will be helpful.

Q2: Another biased random walk

In this random walk, the probabilities are equal, but the step sizes are not: So the probability distribution at time $t + \Delta T$ is related to the probability at time $t$ by convolution with $F_{\Delta T}(x) = \frac{1}{2}(\delta(x - b_-) + \delta(x + b_+))$, where $b_- = b - s$ and $b_+ = b + s$.

A. Calculate $\hat{F}_{\Delta T}(\omega) = \int_{-\infty}^{\infty} F_{\Delta T}(x) e^{-i\omega x} dx$

B. Determine how the probability distribution evolves over time $T$ by determining $\left(\hat{F}_{\Delta T}(\omega)\right)^{T/\Delta T}$ in the limit of $\Delta T \to 0$ with (as in the text) $b^2 = A\Delta T$ but also $s = S\Delta T$.

C. Can this behavior be distinguished from that of the biased random walk in Q1? Why or why not?