Linear Systems, Black Boxes, and Beyond

Homework #3 (2020-2021), Answers

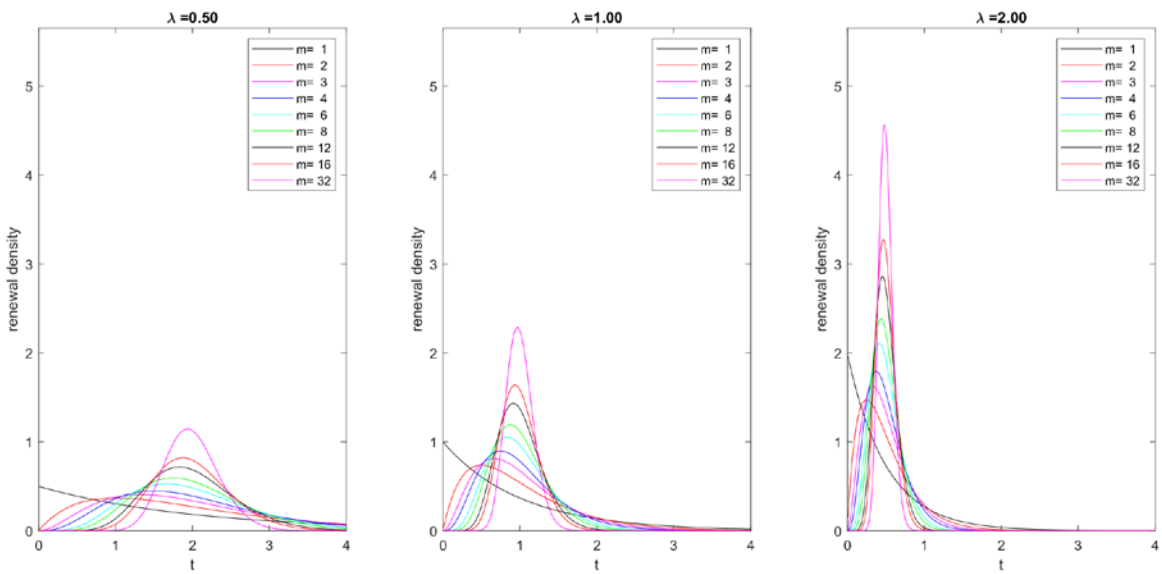
Q1: Spectra of some other renewal processes.

This is a computational exercise about non-Poisson renewal processes. A "gamma process" of order m (here, $m>0$) and rate λ is a renewal process whose renewal density is $g_m(t;\lambda)=\frac{t^{m-1}(\lambda m)^m}{\Gamma(m)}e^{-t\lambda m}$. ($\Gamma(m)$ is the gamma-function, $\Gamma(m)=\int_0^\infty u^{m-1}e^{-u}du$, and $\Gamma(m)=(m-1)!$ for $m=1,2,3,\ldots$). Note that g_m is properly normalized: $\int_0^\infty g_m(t;\lambda)dt=\frac{1}{\Gamma(m)}\int_0^\infty t^{m-1}(\lambda m)^m e^{-t\lambda m}dt=\frac{1}{\Gamma(m)}\int_0^\infty u^{m-1}e^{-u}du=1$, with second step using $u=t\lambda m$. For integers $m\geq 1$, a gamma-process can be derived from a Poisson process of rate λm, by taking every m th event. We don't show this here; see supplementary material below.

What this means is that the renewal density for $g_m(t)$ is the m-fold convolution of the renewal density of a Poisson process of rate $m\lambda$, with itself. Since convolution in the time domain is multiplication in the frequency domain, it follows that $\tilde{g}_m(\omega;\lambda)=\left(\frac{1}{1+\frac{i\omega}{m\lambda}}\right)^m$ -- as shown in the supplementary material below.

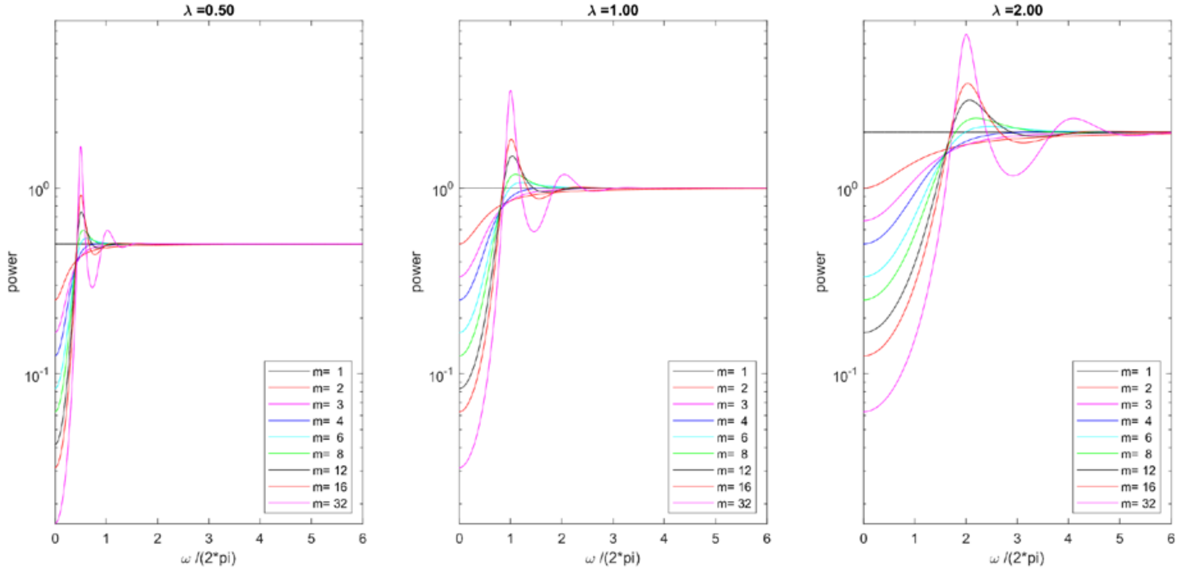
Using this as a starting point:

A: Plot the renewal density of a gamma process of order m, i.e., $g_m(t;\lambda)=\frac{t^{m-1}(\lambda m)^m}{\Gamma(m)}e^{-t\lambda m}$, for a few values of m and λ.



Note that the renewal density becomes progressively more sharply peaked, leading to a progressively more clock-like process.

B. Plot the corresponding power spectra.



C. How do you interpret the behavior of the power spectrum as $\omega \to 0$ and as $\omega \to \infty$?
At low frequencies, the power spectrum is proportional to λ / m: as it becomes more regular (increasing *m*), there is less variation over short intervals. At high frequencies, the power spectrum becomes proportional to the rate; the flatness of the spectrum indicates that there at fast timescales, there is no structure in the point process.
D. For what value of m does the power spectrum first have a peak at a nonzero frequency?
$m = 4$ has the very slightest peak. See table of max values of power spectra, below.

```
spec_renew_gammaproc_demo
lambda
    0.5000    1.0000    2.0000
    m     max of spectra
    1.0000    0.5000    1.0000    2.0000
    2.0000    0.4994    0.9950    1.9615
    3.0000    0.5000    0.9998    1.9943
    4.0000    0.5028    1.0055    2.0110
    6.0000    0.5360    1.0720    2.1440
    8.0000    0.5940    1.1880    2.3759
   12.0000    0.7445    1.4890    2.9780
   16.0000    0.9170    1.8340    3.6679
   32.0000    1.6768    3.3536    6.7071
    m     min of spectra
    1.0000    0.5000    1.0000    2.0000
    2.0000    0.2500    0.5000    1.0000
    3.0000    0.1667    0.3333    0.6667
    4.0000    0.1250    0.2500    0.5000
    6.0000    0.0833    0.1667    0.3333
    8.0000    0.0625    0.1250    0.2500
   12.0000    0.0417    0.0833    0.1667
   16.0000    0.0313    0.0625    0.1250
   32.0000    0.0156    0.0313    0.0625
diary off
```

```
% spec_renew_gammaproc_demo: show the spectrum and renewal density of gamma processes
%
if ~exist('dw') dw=0.001; end
if ~exist('wmax') wmax=40; end
if ~exist('m_list') m_list=[1 2 3 4 6 8 12 16 32]; end
if ~exist('lambda_list') lambda_list=[0.5 1 2]; end
if ~exist('dt') dt=0.01; end
if ~exist('tmax') tmax=4; end
```

```
if ~exist('colors') colors='krmbcg';end
t=[dt:dt:tmax];
w=[0:dw:wmax];
%
%plot renewal density
%
figure;
set(gcf,'Position',[100 100 1400 600]);
for il=1:length(lambda_list)
    ls=[];
    hl=[];
    lambda=lambda_list(il);
    subplot(1,length(lambda_list),il);
    for im=1:length(m_list)
        c=colors(mod(im-1,length(colors))+1);
        m=m_list(im);
        g=(((m*lambda).^m)/gamma(m))*t.^(m-1).*exp(-t*lambda*m);
        hl(im,1)=plot(t,g,c);
        hold on;
        ls=strvcat(ls,sprintf('m=%3.0f',m));
    end %im
    xlabel('t');
    set(gca,'XLim',[0 max(t)]);
    set(gca,'YLim',[0 max(lambda_list*sqrt(max(m)))/2]);
    ylabel('renewal density');
    legend(hl,ls,'Location','NorthEast');
    title(cat(2,'\lambda ',sprintf('=%4.2f',lambda)));
end %il
%
%plot power spectra
%
figure;
set(gcf,'Position',[100 100 1400 600]);
maxspec=zeros(length(m_list),length(lambda_list));
minspec=zeros(length(m_list),length(lambda_list));
for il=1:length(lambda_list)
    ls=[];
    hl=[];
    lambda=lambda_list(il);
    subplot(1,length(lambda_list),il);
    for im=1:length(m_list)
        c=colors(mod(im-1,length(colors))+1);
        m=m_list(im);
        pspec=(1./(1+i*w/(m*lambda))).^m;
        s=lambda*(1-abs(pspec).^2)./(abs(1-pspec)).^2;
        hl(im,1)=semilogy(w/(2*pi),s,c);
        hold on;
        ls=strvcat(ls,sprintf('m=%3.0f',m));
        maxspec(im,il)=max(s);
        minspec(im,il)=min(s);

    end %im
    xlabel('\omega /(2*pi)');
    set(gca,'XLim',[0 floor(wmax/(2*pi))]);
    set(gca,'YLim',[min(lambda_list)/max(m_list),4*max(lambda_list)]);
    ylabel('power');
    legend(hl,ls,'Location','SouthEast');
    title(cat(2,'\lambda ',sprintf('=%4.2f',lambda)));
end %il
%max and minima
disp('lambda')
disp(lambda_list);
disp('    m    max of spectra');
```

```
disp([m_list',maxspec]);
disp('    m    min of spectra');
disp([m_list',minspec]);
```

Q2-Q4 concern the "global coherence, which is a kind of generalization of pairwise coherence. See Cimenser et al., "Tracking brain states under general anesthesia using global coherence analysis", PNAS 108, 8832-8837.

Q2: Cross-spectral matrix and global coherence: definition and basic properties

Say we have a set $X_1, X_2, ..., X_N$ *of random signals. Let* $P_{X_j,X_k}(\omega)$ *is the cross-spectrum of* X_j *and* X_k*. The cross-spectral matrix* $M(\omega)$ *is defined as the matrix whose elements* $M_{j,k}(\omega) = P_{X_j,X_k}(\omega)$*. The global coherence at the frequency* ω *is defined as the ratio of the largest eigenvalue of* $M(\omega)$ *to the sum of its eigenvalues.*

A. Is $M(\omega)$ *self-adjoint?*

Yes, because $M_{k,j}(\omega) = P_{X_k,X_j}(\omega) = P_{X_j,X_k}(-\omega) = \overline{P_{X_j,X_k}(\omega)} = \overline{M_{j,k}(\omega)}$.

B. Part A means that the eigenvalues of $M(\omega)$ *are real. Here we show that they also must be non-negative.*

First, show that if a matrix A *has the property that* z^*Az *is real and non-negative for all vectors* $z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}$

(where z^* *is the conjugate transpose of* z*), then all eigenvalues of* A *are non-negative.*
Then, using the definition of the cross-spectrum in terms of Fourier estimates,

$P_{X,Y}(\omega) = \lim_{T\to\infty} \frac{1}{T}\left\langle F(x,\omega,T,0)\overline{F(y,\omega,T,0)}\right\rangle$, *show that for any vector* $z(\omega) = \begin{pmatrix} z_1(\omega) \\ z_2(\omega) \\ \vdots \\ z_N(\omega) \end{pmatrix}$, *that*

$z^*(\omega)M(\omega)z(\omega) \geq 0$.

First part: Say z is an eigenvector of A with eigenvalue λ. Then $z^*Az = z^*\lambda z = \lambda z^* z = \lambda |z|^2$. Since we are given that $z^*Az \geq 0$, and that $|z| \neq 0$ (definition of an eigenvector), $\lambda = \frac{z^*Az}{|z|^2} \geq 0$

Second part ($z^*(\omega)M(\omega)z(\omega) \geq 0$): In coordinates, we need to show that the following expression is ≥ 0:

$$z^*(\omega)M(\omega)z(\omega) = \sum_{j,k} \left(z^*(\omega)\right)_j M_{j,k}(\omega) z_k(\omega) = \sum_{j,k} \overline{z_j(\omega)} M_{j,k}(\omega) z_k(\omega)$$

Each element $M_{j,k}(\omega)$ of $M(\omega)$ is a limit of Fourier estimates, so

$$z^*(\omega)M(\omega)z(\omega)=\sum_{j,k}\overline{z_j(\omega)}M_{j,k}(\omega)z_k(\omega)=\lim_{T\to\infty}\frac{1}{T}\sum_{j,k}\overline{z_j(\omega)}\left\langle F(x_j,\omega,T,0)\overline{F(x_k,\omega,T,0)}\right\rangle z_k(\omega)$$

$$=\lim_{T\to\infty}\frac{1}{T}\sum_{j,k}\left\langle\overline{z_j(\omega)}F(x_j,\omega,T,0)\overline{F(x_k,\omega,T,0)}\,z_k(\omega)\right\rangle .$$

$$=\lim_{T\to\infty}\frac{1}{T}\left\langle\left(\sum_j\overline{z_j(\omega)}F(x_j,\omega,T,0)\right)\left(\sum_k\overline{F(x_k,\omega,T,0)}\,z_k(\omega)\right)\right\rangle$$

The two terms inside the average $\langle\ \rangle$ are complex-conjugates of each other. So

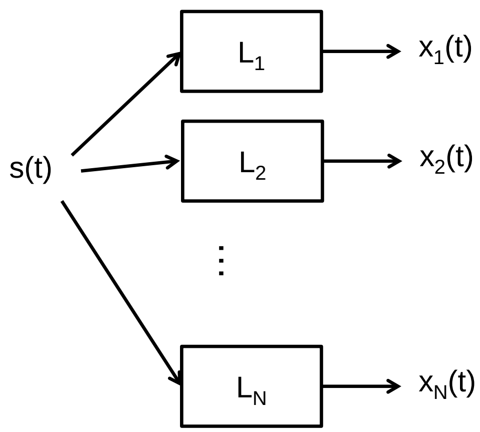
$z^*(\omega)M(\omega)z(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|\left(\sum_j\overline{z_j(\omega)}F(x_j,\omega,T,0)\right)\right|^2\right\rangle$. The right-hand side is a sum of non-negative quantities, so $z^*(\omega)M(\omega)z(\omega)\geq 0$.

C. What is the smallest possible value of the global coherence of N signals?

$1/N$: Since the eigenvalues are non-negative and sum to the trace, the largest eigenvalue must be at least $1/N$ th of the trace.

Q3: Global coherence: a single, common noise source

A. Consider the following system, in which each of the signals X_j are generated by a separate linear filter L_j acting on the same noise input $s(t)$, whose power spectrum is $P_S(\omega)$.. Determine $P_{X_j,X_k}(\omega)$ in terms of and $P_S(\omega)$ and the transfer functions $\tilde{L}_j(\omega)$ and $\tilde{L}_k(\omega)$ of L_j and L_k.



Each output signal $x_j(t)$ is the response of L_j to $s(t)$. So (as in Q2) we can approximate a Fourier estimate for x_j: $F(x_j,\omega,T,0)\approx\tilde{L}_j(\omega)F(s_j,\omega,T,0)$, with the approximation converging for long T. From this,

$$P_{X_j,X_k}(\omega) = \lim_{T\to\infty}\frac{1}{T}\left\langle F(x_j,\omega,T,0)\overline{F(x_k,\omega,T,0)}\right\rangle$$
$$= \lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{L}_j(\omega)F(s,\omega,T,0)\overline{\tilde{L}_k(\omega)}\overline{F(s,\omega,T,0)}\right\rangle$$
$$= \tilde{L}_j(\omega)\overline{\tilde{L}_k(\omega)}\lim_{T\to\infty}\frac{1}{T}\left\langle F(s,\omega,T,0)\overline{F(s,\omega,T,0)}\right\rangle$$
$$= \tilde{L}_j(\omega)\overline{\tilde{L}_k(\omega)}P_S(\omega)$$

, where the first step is the definition of the cross-spectrum, the second step follows from the input-output relationship of the linear filters $\tilde{L}_j(\omega)$ and $\tilde{L}_k(\omega)$, and the final step follows from the definition of the power spectrum of $s(t)$.

B. Show that the vector $v(\omega) = \begin{pmatrix} \tilde{L}_1(\omega) \\ \tilde{L}_2(\omega) \\ \vdots \\ \tilde{L}_N(\omega) \end{pmatrix}$ *is an eigenvector of the cross-spectral matrix* $M(\omega)$*, and find its corresponding eigenvalue.*

We compute $M(\omega)v(\omega)$ in coordinates:

$$\left(M(\omega)v(\omega)\right)_j = \sum_k M_{j,k}(\omega)v_k(\omega) = P_S(\omega)\sum_k \tilde{L}_j(\omega)\overline{\tilde{L}_k(\omega)}\tilde{L}_k(\omega) = P_S(\omega)\tilde{L}_j(\omega)\sum_k \overline{\tilde{L}_k(\omega)}\tilde{L}_k(\omega).$$

So, $M(\omega)v(\omega) = \left(P_S(\omega)\sum_k \overline{\tilde{L}_k(\omega)}\tilde{L}_k(\omega)\right)v(\omega)$, and the eigenvalue is $P_S(\omega)\sum_k \overline{\tilde{L}_k(\omega)}\tilde{L}_k(\omega)$.

C. Show that for the above system, the global coherence is 1.

Method 1. The trace of the cross-spectral matrix is $trM(\omega) = \sum_k M_{k,k}(\omega) = P_S(\omega)\sum_k \overline{\tilde{L}_k(\omega)}\tilde{L}_k(\omega)$, which is the same as the eigenvalue found in part B. Since the trace is the sum of the eigenvalues, and no eigenvalues are negative, the other eigenvalues are zero.

Method 2. We use a different approach to show that $v(\omega)$ is the only eigenvector with a nonzero eigenvalue, without relying on Q2B.

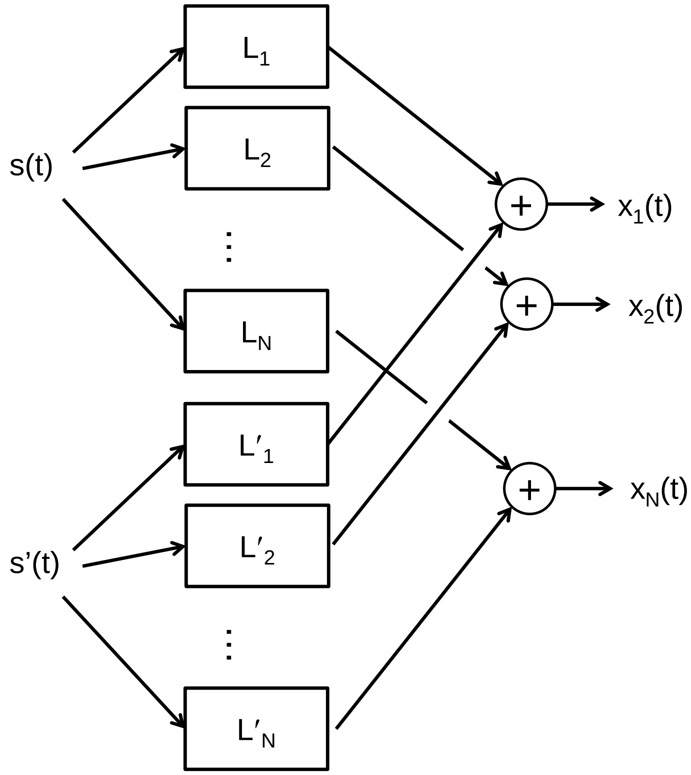
First, we show that for the above cross-spectral matrix and any vector $w(\omega)$, show that $M(\omega)w(\omega)$ is proportional to $v(\omega)$.

$$\left(M(\omega)w(\omega)\right)_j = \sum_k M_{j,k}(\omega)w_k(\omega) = P_S(\omega)\sum_k \tilde{L}_j(\omega)\overline{\tilde{L}_k(\omega)}w_k(\omega) = P_S(\omega)\tilde{L}_j(\omega)\sum_k \overline{\tilde{L}_k(\omega)}w_k(\omega)$$, so

$$M(\omega)w(\omega) = \left(P_S(\omega)\sum_k \overline{\tilde{L}_k(\omega)}w_k(\omega)\right)v(\omega)$$

Then we show that the eigenvector identified in B is the only eigenvector with a nonzero eigenvalue. Since the above cross-spectral matrix is self-adjoint, its eigenvectors form an orthogonal basis. Part B showed that one eigenvector is $v(\omega)$, and it has a nonzero eigenvalue. We need to show that all the other eigenvectors have eigenvalue zero. Say $z(\omega)$ is an eigenvector distinct from $v(\omega)$, with $M(\omega)z(\omega)$ nonzero. Then $M(\omega)z(\omega)$ is proportional to $z(\omega)$ (since $z(\omega)$ is an eigenvector), but also (from D) $M(\omega)z(\omega)$ is proportional to $v(\omega)$, and therefore orthogonal to $z(\omega)$ (since $v(\omega)$ and $z(\omega)$ are both eigenvectors of the self-adjoint $M(\omega)$). This is a contradiction, so $M(\omega)z(\omega)$ must be zero.

Q4. Now consider the following system, where $s(t)$ and $s'(t)$ are independent noises, with power spectra $P_{X_j,X_k}(\omega)$ $P_S(\omega)$ and $P_{S'}(\omega)$; the rest of the set-up is as above.



A. Determine $P_{X_j,X_k}(\omega)$

Each output signal $x_j(t)$ is a sum of two components,one via L_j and one via L'_j. So (as in Q2) we can approximate a Fourier estimate for x_j: $F(x_j,\omega,T,0) \approx \tilde{L}_j(\omega)F(s,\omega,T,0) + \tilde{L}'_j(\omega)F(s',\omega,T,0)$, with the approximation converging for long T. Therefore:

$$P_{X_j,X_k}(\omega) = \lim_{T\to\infty}\frac{1}{T}\left\langle \left(\tilde{L}_j(\omega)F(s,\omega,T,0)+\tilde{L}'_j(\omega)F(s',\omega,T,0)\right)\left(\overline{\tilde{L}_k(\omega)F(s,\omega,T,0)+\tilde{L}'_k(\omega)F(s',\omega,T,0)}\right)\right\rangle.$$

The cross-terms between Fourier estimates for s and s' have an avarage of zero, since s and s' are independent. So,

$$P_{X_j,X_k}(\omega) = \lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{L}_j(\omega)F(s,\omega,T,0)\overline{\tilde{L}_k(\omega)F(s,\omega,T,0)}+\tilde{L}'_j(\omega)F(s',\omega,T,0)\tilde{L}'_k(\omega)F(s',\omega,T,0)\right\rangle$$
$$= \lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{L}_j(\omega)F(s,\omega,T,0)\overline{\tilde{L}_k(\omega)F(s,\omega,T,0)}\right\rangle + \lim_{T\to\infty}\frac{1}{T}\left\langle \tilde{L}'_j(\omega)F(s',\omega,T,0)\tilde{L}'_k(\omega)F(s',\omega,T,0)\right\rangle.$$

Since the L's and L''s are just multipliers, this leads to

$$P_{X_j,X_k}(\omega) = \tilde{L}_j(\omega)\overline{\tilde{L}_k(\omega)}\lim_{T\to\infty}\frac{1}{T}\left\langle F(s,\omega,T,0)\overline{F(s,\omega,T,0)}\right\rangle + \tilde{L}'_j(\omega)\overline{\tilde{L}'_k(\omega)}\lim_{T\to\infty}\frac{1}{T}\left\langle F(s',\omega,T,0)F(s',\omega,T,0)\right\rangle$$
$$= \tilde{L}_j(\omega)\overline{\tilde{L}_k(\omega)}P_S(\omega) + \tilde{L}'_j(\omega)\overline{\tilde{L}'_k(\omega)}P_{S'}(\omega).$$

where the last step follows from the relationship of the power spectrum to the Fourier estimates for s and s'.

B. Show that the range of the cross-spectral matrix $M(\omega)$ is of dimension at most 2.

For any vector $w(\omega)$, we calculate $M(\omega)w(\omega)$ in coordinates:

$$\left(M(\omega)w(\omega)\right)_j = \sum_k M_{j,k}(\omega)w_k(\omega) = \left(P_S(\omega)\sum_k \tilde{L}_j(\omega)\overline{\tilde{L}_k(\omega)} + P_{S'}(\omega)\sum_k \tilde{L}'_j(\omega)\overline{\tilde{L}_k{}'(\omega)}\right)w_k(\omega)$$

$$= \tilde{L}_j(\omega)\left(P_S(\omega)\sum_k \overline{\tilde{L}_k(\omega)}w_k(\omega)\right) + \tilde{L}'_j(\omega)\left(P_{S'}(\omega)\sum_k \overline{\tilde{L}'_k(\omega)}w_k(\omega)\right).$$

This exhibits $M(\omega)w(\omega)$ as a linear combination of the two column vectors $v(\omega) = \begin{pmatrix} \tilde{L}_1(\omega) \\ \tilde{L}_2(\omega) \\ \vdots \\ \tilde{L}_N(\omega) \end{pmatrix}$ and

$v'(\omega) = \begin{pmatrix} \tilde{L}'_1(\omega) \\ \tilde{L}'_2(\omega) \\ \vdots \\ \tilde{L}'_N(\omega) \end{pmatrix}$*.We can write this more compactly as*

$$M(\omega)w(\omega) = v(\omega)P_S(\omega)\left(v(\omega)^* w(\omega)\right) + v'(\omega)P_{S'}(\omega)\left(v'(\omega)^* w(\omega)\right).$$

C. How many nonzero eigenvalues does $M(\omega)$ have?Under what circumstances is the global coherence equal to 1?

It can have at most two nonzero eigenvalues, since any eigenvector with a nonzero eigenvalue must be a linear combination of $v(\omega)$ and $v'(\omega)$ above. Conversely, if these two vectors are linearly independent, then there must be two vectors with nonzero eigenvalues. Assume to the contrary – that the only vector with a nonzero eigenvalue was some y. If so, the range of $M(\omega)$ would have to be one-dimensional, namely, scalar multiple of y. But this leads to a contradiction, using the final result of part B. Starting with v, find a scalar α such that $w = v - \alpha v'$ is orthogonal to v'. Then, the final result of part B shows that $M(\omega)w(\omega)$ is a multiple of v, since the second term is zero. Similarly, starting with v', find a scalar α' such that $w' = v' - \alpha' v$ is orthogonal to v. Then, the final result of part B shows that $M(\omega)w'(\omega)$ is a multiple of v'. This means that the range of $M(\omega)$ is at least two-dimensional.

If the global coherence is 1, $M(\omega)$ has only one eigenvector with a nonzero eigenvalue, and, by the above argument, of $v(\omega)$ and $v'(\omega)$ must be scalar multiples of each other, i.e., there's a constant C for which $\dfrac{L_k(\omega)}{L'_k(\omega)} = C$ for all k.

Supplementary material for Q1 about gamma processes

Here we determine the Fourier transform of the renewal density of a gamma process. We do this by finding the renewal density of a gamma process of order m and rate λ / m (rather than rate λ), since – as the calculation will show – this is the m-fold convolution of the renwal density of a Poisson process of rate λ. That is, we determine the Fourier transform of $s_m(t;\lambda) = g_m(t;\lambda / m) = \dfrac{t^{m-1}\lambda^m}{\Gamma(m)}e^{-t\lambda}$.

We do this via a method, "generating functions", that is widely useful, produces the answer for all m at once. The idea is that we look at $S(t,y;\lambda) = \sum_{m=1}^{\infty} y^{m-1}s_m(t;\lambda)$, and compute its Fourier transform. Since

$\tilde{S}(\omega, y; \lambda) = \int_0^\infty S(t, y; \lambda) e^{-i\omega t} dt = \int_0^\infty \sum_{m=1}^\infty y^{m-1} s_m(t; \lambda) dt = \sum_{m=1}^\infty y^{m-1} \tilde{s}_m(\omega; \lambda)$, we can then pull out the terms involving y^{m-1} in $\tilde{S}(\omega, y; \lambda)$ to get the Fourier transform $\tilde{s}_m(\omega; \lambda)$ of $s_m(t; \lambda)$.

The generating-function method works because $S(t, y; \lambda)$ has a nice form:

$$S(t, y; \lambda) = \sum_{m=1}^\infty y^{m-1} s_m(t; \lambda) = \sum_{m=1}^\infty y^{m-1} \frac{t^{m-1}\lambda^m}{\Gamma(m)} e^{-t\lambda} = \lambda e^{-t\lambda} \sum_{m=1}^\infty \frac{y^{m-1} t^{m-1} \lambda^{m-1}}{\Gamma(m)}$$
$$= \lambda e^{-t\lambda} \sum_{n=0}^\infty \frac{(yt\lambda)^n}{\Gamma(n+1)} = \lambda e^{-t\lambda} \sum_{n=0}^\infty \frac{(yt\lambda)^n}{n!} = \lambda e^{-t\lambda + yt\lambda}.$$

So,

$$\tilde{S}(\omega, y; \lambda) = \int_0^\infty S(t, y; \lambda) e^{-i\omega t} dt = \int_0^\infty \lambda e^{-t\lambda + yt\lambda} e^{-i\omega t} dt$$
$$= \lambda \frac{1}{-\lambda + y\lambda - i\omega} e^{-t\lambda + yt\lambda} e^{-i\omega t} \Big|_0^\infty = \lambda \frac{1}{\lambda + i\omega - y\lambda} = \frac{\lambda}{\lambda + i\omega} \frac{1}{1 - \frac{\lambda y}{\lambda + i\omega}}.$$

Note that the final expression is of the form $a \frac{1}{1 - ry}$, the sum of a geometric series whose n th term is $ar^n y^n$.

So the term involving y^{m-1} is $\frac{\lambda}{\lambda + i\omega} \left(\frac{\lambda}{\lambda + i\omega} \right)^{m-1} y^{m-1} = \left(\frac{\lambda}{\lambda + i\omega} \right)^m y^{m-1}$. So, the Fourier transform $\tilde{s}_m(\omega; \lambda)$ of $s_m(t; \lambda)$ is the coefficient of y^{m-1} in this term, namely, $\left(\frac{\lambda}{\lambda + i\omega} \right)^m$. So $\tilde{s}_m(\omega; \lambda) = \left(\frac{\lambda}{\lambda + i\omega} \right)^m = \left(\frac{1}{1 + \frac{i\omega}{\lambda}} \right)^m$, corresponding to the m -fold convolution of the Poisson renewal density with itself.

Finally,

$$g_m(t; \lambda) = s_m(t; m\lambda), \text{ so } \tilde{g}_m(\omega; \lambda) = \tilde{s}_m(\omega; m\lambda) = \left(\frac{1}{1 + \frac{i\omega}{m\lambda}} \right)^m.$$