Linear Systems, Black Boxes, and Beyond

Homework #3 (2020-2021), Questions

Q1: Spectra of some other renewal processes

This is a computational exercise about non-Poisson renewal processes. A “gamma process” of order \( m \) (here, \( m > 0 \)) and rate \( \lambda \) is a renewal process whose renewal density is 
\[
g_m(t; \lambda) = \frac{t^{m-1}(\lambda t)^m}{\Gamma(m)} e^{-\lambda t}.
\]
\((\Gamma(m)\) is the gamma-function, \( \Gamma(m) = \int_0^\infty u^{m-1} e^{-u} du \), and \( \Gamma(m) = (m-1)! \) for \( m = 1, 2, 3, \ldots \)). Note that \( g_m \) is properly normalized:
\[
\int_0^\infty g_m(t; \lambda) dt = \frac{1}{\Gamma(m)} \int_0^\infty t^{m-1}(\lambda t)^m e^{-t \lambda} dt = \frac{1}{\Gamma(m)} \int_0^\infty u^{m-1} e^{-u} du = 1,
\]
with second step using \( u = t \lambda m \). For integers \( m \geq 1 \), a gamma-process can be derived from a Poisson process of rate \( \lambda m \) by taking every \( m \) th event. We don’t show this here; see supplementary material below.

What this means is that the renewal density for \( g_m(t) \) is the \( m \)-fold convolution of the renewal density of a Poisson process of rate \( m \lambda \), with itself. Since convolution in the time domain is multiplication in the frequency domain, it follows that
\[
\hat{g}_m(\omega; \lambda) = \left( \frac{1}{1 + \frac{i \omega}{m \lambda}} \right)^m
\]
-- as shown in the supplementary material below.

Using this as a starting point:

A: Plot the renewal density of a gamma process of order \( m \), i.e., 
\( g_m(t; \lambda) = \frac{t^{m-1}(\lambda t)^m}{\Gamma(m)} e^{-\lambda t} \), for a few values of \( m \) and \( \lambda \).
B. Plot the corresponding power spectra.
C. How do you interpret the behavior of the power spectrum as \( \omega \to 0 \) and as \( \omega \to \infty \)?
D. For what value of \( m \) does the power spectrum first have a peak at a nonzero frequency?

Q2-Q4 concern the “global coherence, which is a kind of generalization of pairwise coherence. See Cimenser et al., “Tracking brain states under general anesthesia using global coherence analysis”, PNAS 108, 8832-8837.

Q2: Cross-spectral matrix and global coherence: definition and basic properties

Say we have a set \( X_1, X_2, \ldots, X_N \) of random signals. Let \( P_{X_j, X_k}(\omega) \) is the cross-spectrum of \( X_j \) and \( X_k \). The cross-spectral matrix \( M(\omega) \) is defined as the matrix whose elements \( M_{j,k}(\omega) = P_{X_j, X_k}(\omega) \). The global coherence at the frequency \( \omega \) is defined as the ratio of the largest eigenvalue of \( M(\omega) \) to the sum of its eigenvalues.
A. Is \( M(\omega) \) self-adjoint?
B. Part A means that the eigenvalues of \( M(\omega) \) are real. Here we show that they also must be non-negative.
First, show that if a matrix $A$ has the property that $z^*Az$ is real and non-negative for all vectors $z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}$ (where $z^*$ is the conjugate transpose of $z$), then all eigenvalues of $A$ are non-negative.

Then, using the definition of the cross-spectrum in terms of Fourier estimates, 

$$P_{x,y}(\omega) = \lim_{t \to -\infty} \frac{1}{T} \left\{ F(x, \omega, T, 0) \overline{F(y, \omega, T, 0)} \right\},$$ 

show that for any vector $z(\omega) = \begin{pmatrix} z_1(\omega) \\ z_2(\omega) \\ \vdots \\ z_N(\omega) \end{pmatrix}$, that 

$$z^*(\omega)M(\omega)z(\omega) \geq 0.$$ 

C. What is the smallest possible value of the global coherence of $N$ signals?

$1/N$ : Since the eigenvalues are non-negative and sum to the trace, the largest eigenvalue must be at least $1/N$ th of the trace.

Q3: Global coherence: a single, common noise source

A. Consider the following system, in which each of the signals $X_j$ are generated by a separate linear filter $L_j$ acting on the same noise input $s(t)$, whose power spectrum is $P_s(\omega)$. Determine $P_{X_j,X_k}(\omega)$ in terms of $P_s(\omega)$ and the transfer functions $\tilde{L}_j(\omega)$ and $\tilde{L}_k(\omega)$ of $L_j$ and $L_k$.

B. Show that the vector $v(\omega) = \begin{pmatrix} \tilde{L}_1(\omega) \\ \tilde{L}_2(\omega) \\ \vdots \\ \tilde{L}_N(\omega) \end{pmatrix}$ is an eigenvector of the cross-spectral matrix $M(\omega)$, and find its corresponding eigenvalue.
Q4. Now consider the following system, where $s(t)$ and $s'(t)$ are independent noises, with power spectra $P_{s_j,s_j}(\omega)$, $P_s(\omega)$ and $P_{s'}(\omega)$; the rest of the set-up is as above.

A. Determine $P_{s_j,s_j}(\omega)$

B. Show that the range of the cross-spectral matrix $M(\omega)$ is of dimension at most 2.
Supplementary material for Q1 about gamma processes

Here we determine the Fourier transform of the renewal density of a gamma process. We do this by finding the renewal density of a gamma process of order \(m\) and rate \(\lambda/m\) (rather than rate \(\lambda\)), since – as the calculation will show – this is the \(m\)-fold convolution of the renewal density of a Poisson process of rate \(\lambda\). That is, we determine the Fourier transform of \(s_m(t; \lambda) = g_m(t; \lambda/m) = \frac{t^{m-1} \lambda^m}{\Gamma(m)} e^{-t\lambda}\).

We do this via a method, “generating functions”, that is widely useful, produces the answer for all \(m\) at once.

The idea is that we look at \(S(t, y; \lambda) = \sum_{m=1}^{\infty} y^{m-1}s_m(t; \lambda)\), and compute its Fourier transform. Since

\[
\tilde{S}(\omega, y; \lambda) = \int_{0}^{\infty} S(t, y; \lambda) e^{-i\omega t} dt = \int_{0}^{\infty} \sum_{m=1}^{\infty} y^{m-1}s_m(t; \lambda) dt = \sum_{m=1}^{\infty} y^{m-1}\tilde{s}_m(\omega; \lambda),
\]

we can then pull out the terms involving \(y^{m-1}\) in \(\tilde{S}(\omega, y; \lambda)\) to get the Fourier transform \(\tilde{s}_m(\omega; \lambda)\) of \(s_m(t; \lambda)\).

The generating-function method works because \(S(t, y; \lambda)\) has a nice form:

\[
S(t, y; \lambda) = \sum_{m=1}^{\infty} y^{m-1}s_m(t; \lambda) = \sum_{m=1}^{\infty} y^{m-1} \frac{t^{m-1} \lambda^m}{\Gamma(m)} e^{-t\lambda} = \lambda e^{-t\lambda} \sum_{m=1}^{\infty} \frac{y^{m-1}t^{m-1} \lambda^{m-1}}{\Gamma(m)}
\]

\[
= \lambda e^{-t\lambda} \sum_{n=0}^{\infty} \frac{(yt\lambda)^n}{n!} = \lambda e^{-t\lambda + yt\lambda}
\]

So,

\[
\tilde{S}(\omega, y; \lambda) = \int_{0}^{\infty} S(t, y; \lambda) e^{-i\omega t} dt = \int_{0}^{\infty} \lambda e^{-t\lambda + yt\lambda} e^{-i\omega t} dt
\]

\[
= \frac{\lambda}{-\lambda + y\lambda - i\omega} e^{-t\lambda + yt\lambda} e^{-i\omega t} \bigg|_{0}^{\infty} = \frac{\lambda}{\lambda + i\omega - y\lambda} = \frac{\lambda}{\lambda + i\omega} \frac{1}{1 - \frac{y}{\lambda}}.
\]

Note that the final expression is of the form \(a \frac{1}{1 - ry}\), the sum of a geometric series whose \(n\)th term is \(ar^n y^n\).

So the term involving \(y^{m-1}\) is \(\frac{\lambda}{\lambda + i\omega} \left(\frac{\lambda}{\lambda + i\omega}\right)^{m-1} y^{m-1} = \left(\frac{\lambda}{\lambda + i\omega}\right)^m y^{m-1}\). So, the Fourier transform \(\tilde{s}_m(\omega; \lambda)\) of \(s_m(t; \lambda)\) is the coefficient of \(y^{m-1}\) in this term, namely, \(\left(\frac{\lambda}{\lambda + i\omega}\right)^m\). So \(\tilde{s}_m(\omega; \lambda) = \left(\frac{\lambda}{\lambda + i\omega}\right)^m = \left(\frac{1}{1 + \frac{i\omega}{\lambda}}\right)^m\), corresponding to the \(m\)-fold convolution of the Poisson renewal density with itself.

Finally,

\[
g_m(t; \lambda) = s_m(t; m\lambda), \quad \tilde{g}_m(\omega; \lambda) = \tilde{s}_m(\omega; m\lambda) = \left(\frac{1}{1 + \frac{i\omega}{m\lambda}}\right)^m.
\]