Q1: Eigenvalues and eigenvectors of a rotation matrix. Let \( A = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \). Use the characteristic equation to find its eigenvalues, and then find its eigenvectors.

Eigenvalues: For the above \( A \), \( \det(zI - A) = 0 \) corresponds to \( \det \begin{pmatrix} z - \cos \theta & -\sin \theta \\ \sin \theta & z - \cos \theta \end{pmatrix} = 0 \), i.e.,

\[
(z - \cos \theta)^2 - (\sin \theta)(\sin \theta) = 0,
\]
which simplifies to \( z^2 - 2z \cos \theta + 1 = 0 \). Solving for \( z \) via the quadratic formula,

\[
z = \frac{2 \cos \theta \pm \sqrt{4 \cos^2 \theta - 4}}{2} = \cos \theta \pm \sqrt{-\sin^2 \theta} = \cos \theta \pm i \sin \theta = e^{i \theta}.
\]

Eigenvectors: Say \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) is an eigenvector corresponding to the eigenvalue \( e^{i \theta} \). Then

\[
\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{i \theta} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ i.e.,\ \begin{pmatrix} x_1 \cos \theta + x_2 \sin \theta \\ -x_1 \sin \theta + x_2 \cos \theta \end{pmatrix} = \begin{pmatrix} e^{i \theta} x_1 \\ e^{i \theta} x_2 \end{pmatrix}.
\]
So we need to find solutions to

\[
\begin{cases}
x_1 (\cos \theta - e^{i \theta}) + x_2 \sin \theta = 0 \\
-x_1 \sin \theta + x_2 (\cos \theta - e^{i \theta}) = 0
\end{cases}
\]
or (with \( e^{i \theta} = \cos \theta + i \sin \theta \), to

\[
\begin{cases}
-x_1 \sin \theta + x_2 \sin \theta = 0 \\
x_1 \sin \theta - x_2 \sin \theta = 0
\end{cases}
\]
which reduces to

\[
\begin{cases}
-x_1 + x_2 = 0 \\
x_1 - ix_2 = 0
\end{cases}
\]
This is a degenerate homogeneous system (it had to be – since if \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) is an eigenvector, then so would be any scalar multiple \( \begin{pmatrix} bx_1 \\ bx_2 \end{pmatrix} \)); the equations are satisfied for any \( x_2 = ix_1 \). So \( b \begin{pmatrix} 1 \\ i \end{pmatrix} \) are eigenvectors corresponding to the eigenvalue \( e^{i \theta} \). Similarly, So \( b \begin{pmatrix} 1 \\ -i \end{pmatrix} \) are eigenvectors corresponding to the eigenvalue \( e^{-i \theta} \).

Q2: Eigenvectors and eigenvalues of permutation matrices.

A. Cyclic permutation matrices. Let \( A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \). Note that \( A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_1 \end{pmatrix} \), i.e., \( Ax \) permutes the entries of the vector \( x \). Use this to write the five (very simple) equations corresponding to \( Ax = \lambda x \), and thereby find the eigenvalues and eigenvectors of \( A \).
\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
\end{pmatrix} = \lambda
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
\end{pmatrix}
\]
means \(x_2 = \lambda x_1, x_3 = \lambda x_2, x_4 = \lambda x_3, x_5 = \lambda x_4, x_i = \lambda x_i\). Back-substituting,

\[
x_1 = \lambda x_1 = \lambda^2 x_2 = \lambda^3 x_3 = \lambda^4 x_2 = \lambda^5 x_1.
\]
So, \(x_1 = \lambda^5 x_1\). Since \(x_i\) must be nonzero (otherwise all \(x_j\) would be zero) then \(\lambda^5 = 1\). This means that \(\lambda = e^{\frac{2\pi i}{5}}\) for any integer \(k\). We get distinct eigenvalues for

\[
k \in \{0, 1, 2, 3, 4\}, \text{ and for each such } \lambda_k, \text{ the eigenvectors are } b
\begin{pmatrix}
  1 \\
  e^{\frac{2\pi i}{5} k} \\
  e^{\frac{2\pi i}{5} k} \\
  e^{\frac{2\pi i}{5} k} \\
  e^{\frac{2\pi i}{5} k}
\end{pmatrix}
\]

B. More general permutation matrices. Same as part A, but with

\[
A = \begin{pmatrix}
  0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

Here, \(A x_3 = x_1\), so \(A x_3 = \lambda x_3\) means \(x_2 = \lambda x_1, x_3 = \lambda x_2, x_i = \lambda x_i, x_4 = \lambda x_4, x_5 = \lambda x_5\). Back-substituting, this breaks into two equations: \(x_1 = \lambda x_3 = \lambda^2 x_2 = \lambda^3 x_1\), i.e., \(x_1 = \lambda^3 x_1\), and \(x_4 = \lambda x_5 = \lambda^2 x_4\), i.e., \(x_4 = \lambda^2 x_4\). To ensure that the eigenvector has at least one nonzero coordinate, we need \(x_i \neq 0\) or \(x_i \neq 0\).

If \(x_i \neq 0\), then \(\lambda^3 = 1\) so \(\lambda = e^{\frac{2\pi i}{3}}\), with distinct eigenvalues for \(k \in \{0, 1, 2\}\), and for each such \(\lambda_k\), the eigenvectors are

\[
b \begin{pmatrix}
  1 \\
  e^{\frac{2\pi i}{3} k} \\
  e^{\frac{2\pi i}{3} k} \\
  0 \\
  0
\end{pmatrix}
\]
If \( x_4 \neq 0 \), then \( \lambda^2 = 1 \) so \( \lambda = e^{\frac{2\pi i k}{2}} \), with distinct eigenvalues (+1 and -1) for \( k \in \{0,1\} \), and for each such \( \lambda_k \), the eigenvectors are 
\[
\begin{pmatrix}
0 \\
0 \\
+1 \\
+1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 \\
0 \\
+1 \\
+1
\end{pmatrix}.
\]

If both \( x_i \neq 0 \) and \( x_4 \neq 0 \) are nonzero, then we need a value of \( \lambda \) that satisfies both \( \lambda^3 = 1 \) and \( \lambda^2 = 1 \). This forces \( \lambda = 1 \), and we get eigenvectors that are linear combinations of the \( k = 0 \)-solutions above, namely,
\[
\begin{pmatrix}
1 \\
1 \\
b + c \\
0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 \\
0 \\
1 \\
1
\end{pmatrix}.
\]

In general, any permutation breaks up into disjoint cycles, and a cycle of length \( k \) will lead to an equation like \( \lambda^k = 1 \), and a set of eigenvalues that is nonzero on that cycle.