Linear Transformations and Group Representations

Homework #1 (2020-2021), Answers

Q1: Eigenvalues and eigenvectors of a rotation matrix. Let $A=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. *Use the characteristic equation to find its eigenvalues, and then find its eigenvectors.*

Eigenvalues: For the above A, $\det(zI-A)=0$ corresponds to $\det\begin{pmatrix} z-\cos\theta & -\sin\theta \\ \sin\theta & z-\cos\theta \end{pmatrix}=0$, i.e., $(z-\cos\theta)^2-(-\sin\theta)(\sin\theta)=0$, which simplifies to $z^2-2z\cos\theta+1=0$. Solving for z via the quadratic formula, $z=\dfrac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}=\cos\theta\pm\sqrt{-\sin^2\theta}=\cos\theta\pm i\sin\theta=e^{\pm i\theta}$.

Eigenvectors: Say $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $e^{i\theta}$. Then $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=e^{i\theta}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, i.e., $\begin{pmatrix} x_1\cos\theta+x_2\sin\theta \\ -x_1\sin\theta+x_2\cos\theta \end{pmatrix}=\begin{pmatrix} e^{i\theta}x_1 \\ e^{i\theta}x_2 \end{pmatrix}$. So we need to find solutions to $\begin{cases} x_1\left(\cos\theta-e^{i\theta}\right)+x_2\sin\theta=0 \\ -x_1\sin\theta+x_2\left(\cos\theta-e^{i\theta}\right)=0 \end{cases}$, or (with $e^{i\theta}=\cos\theta+i\sin\theta$), to $\begin{cases} -ix_1\sin\theta+x_2\sin\theta=0 \\ -x_1\sin\theta-ix_2\sin\theta=0 \end{cases}$, which reduces to $\begin{cases} -ix_1+x_2=0 \\ -x_1-ix_2=0 \end{cases}$. This is a degenerate homogeneous system (it had to be – since if $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is an eigenvector, then so would be any scalar multiple $\begin{pmatrix} bx_1 \\ bx_2 \end{pmatrix}$); the equations are satisfied for any $x_2=ix_1$. So $b\begin{pmatrix} 1 \\ i \end{pmatrix}$ are eigenvectors corresponding to the eigenvalue $e^{i\theta}$. Similarly, So $b\begin{pmatrix} 1 \\ -i \end{pmatrix}$ are eigenvectors corresponding to the eigenvalue $e^{-i\theta}$.

Q2: Eigenvectors and eigenvalues of permutation matrices.

A. Cyclic permutation matrices. Let $A=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. *Note that* $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_1 \end{pmatrix}$, *i.e.,* Ax *permutes the entries of the vector* x. *Use this to write the five (very simple) equations corresponding to* $Ax=\lambda x$, *and thereby find the eigenvalues and eigenvectors of* A.

$A\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\lambda\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$ means $x_2=\lambda x_1$, $x_3=\lambda x_2$, $x_4=\lambda x_3$, $x_5=\lambda x_4$, $x_1=\lambda x_5$. Back-substituting, $x_1=\lambda x_5=\lambda^2 x_4=\lambda^3 x_3=\lambda^4 x_2=\lambda^5 x_1$. So, $x_1=\lambda^5 x_1$. Since x_1 must be nonzero (otherwise all x_j would be zero) then $\lambda^5=1$. This means that $\lambda=e^{\frac{2\pi i}{5}k}$ for any integer k. We get distinct eigenvalues for $k\in\{0,1,2,3,4\}$, and for each such λ_k, the eigenvectors are $b\begin{pmatrix}1\\e^{\frac{2\pi i}{5}k}\\e^{\frac{2\pi i}{5}2k}\\e^{\frac{2\pi i}{5}3k}\\e^{\frac{2\pi i}{5}4k}\end{pmatrix}$.

B. More general permutation matrices. Same as part A, but with $A=\begin{pmatrix}0&1&0&0&0\\0&0&1&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix}$.

Here, $A\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}x_2\\x_3\\x_1\\x_5\\x_4\end{pmatrix}$, so $A\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\lambda\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$ means $x_2=\lambda x_1$, $x_3=\lambda x_2$, $x_1=\lambda x_3$, $x_5=\lambda x_4$, $x_4=\lambda x_5$. Back-substituting, this breaks into two equations: $x_1=\lambda x_3=\lambda^2 x_2=\lambda^3 x_1$, i.e., $x_1=\lambda^3 x_1$, and $x_4=\lambda x_5=\lambda^2 x_4$, i.e., $x_4=\lambda^2 x_4$. To ensure that the eigenvector has at least one nonzero coordinate, we need $x_1\neq 0$ or $x_4\neq 0$.

If $x_1\neq 0$, then $\lambda^3=1$ so $\lambda=e^{\frac{2\pi i}{3}k}$, with distinct eigenvalues for $k\in\{0,1,2,\}$, and for each such λ_k, the eigenvectors are $b\begin{pmatrix}1\\e^{\frac{2\pi i}{3}k}\\e^{\frac{2\pi i}{3}2k}\\0\\0\end{pmatrix}$.

If $x_4 \neq 0$, then $\lambda^2 = 1$ so $\lambda = e^{\frac{2\pi i}{2}k}$, with distinct eigenvalues ($+1$ and -1) for $k \in \{0,1\}$, and for each such λ_k, the eigenvectors are $c\begin{pmatrix} 0 \\ 0 \\ 0 \\ +1 \\ +1 \end{pmatrix}$ and $c\begin{pmatrix} 0 \\ 0 \\ 0 \\ +1 \\ -1 \end{pmatrix}$.

If both $x_1 \neq 0$ and $x_4 \neq 0$ are nonzero, then we need a value of λ that satisfies both $\lambda^3 = 1$ and $\lambda^2 = 1$. This forces $\lambda = 1$, and we get eigenvectors that are linear combinations of the $k = 0$-solutions above, namely,

$$b\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

In general, any permutation breaks up into disjoint cycles, and a cycle of length k will lead to an equation like $\lambda^k = 1$, and a set of eigenvalues that is nonzero on that cycle.