Linear Transformations and Group Representations

Homework #2 (2020-2021), Answers for Q2 only

Q2. A five-dimensional space associated with symmetric tensors in 3 dimensions.

This has a long setup and is a bit heavy on algebra, but I think it's worthwhile to how the machinery works, and, it will also serve as an example later on for how group representations work when the set of transformations is not commutative. This question shows how the rotations act on the symmetric tensors in 3-space: that there is a one-dimensional subspace that is invariant when the 3-d coordinates are rotated, and it will demonstrate the way that the coordinate rotations act on the other 5 dimensions.

The setup is the tensor products q of elements in an n-dimensional vector space V, in which we've chosen basis vectors $x_1, x_2, \ldots, x_n$. As we've seen, the tensor-product space $V \otimes V$ has n^2 dimensions as it has a basis consisting of the $x_i \otimes x_j$, and typical members of $V \otimes V$ can be written as $q = \sum_{i.j=1}^{n} q_{i,j} x_i \otimes x_j$. We've seen that an isomorphism A of V yields an isomorphism Ψ_A in $V \otimes V$, namely, $\Psi_A(x_i \otimes x_j) = Ax_i \otimes Ax_j$. We've also seen that Ψ_A maps the symmetric part of $V \otimes V$ into itself, showing that it is an intrinsic aspect of the structure of $V \otimes V$. A basis for the symmetric part can be found by symmetrizing the basis for $V \otimes V$, and consists of $x_i \otimes x_i$ and $\frac{1}{2}\left(x_i \otimes x_j + x_j \otimes x_i\right)$ for $i \neq j$. This also showed that the symmetric part of $V \otimes V$ has $\frac{n(n+1)}{2}$ dimensions.

As is suggested by the example of the diffusion tensor, it is convenient to think of symmetric tensor products as quadratics, i.e., to think of $x_i \otimes x_i$ as x_i^2, and to think of $\frac{1}{2}\left(x_i \otimes x_j + x_j \otimes x_i\right)$ as $x_i x_j$ -- because they transform in the same way.

We now specialize the above picture two ways. First – and this is just to make things more concrete – we set $n = 3$. But also, we only consider the isomorphisms R of V that preserve the length: $\Psi_R(x_1^2 + x_2^2 + x_3^2) = x_1^2 + x_2^2 + x_3^2$. This is equivalent to saying that V has Hilbert space structure and R preserves the dot-product: $\langle Rx, Ry \rangle = \langle x, y \rangle$. We now have a scenario in which Ψ_R acts in a 6-dimensional space (the quadratic polynomials in x_1, x_2, and x_3), and preserves a one-dimensional subspace of it, namely, scalar multiples of $x_1^2 + x_2^2 + x_3^2$. So, complementary to this one-dimensional subspace, there must be a 5-dimensional subspace in which Ψ_R acts non-trivially – and the goal here is examine this action.

We choose the following basis for $sym(V \otimes V)$:

$$r = x_1^2 + x_2^2 + x_3^2$$
$$s = \frac{1}{\sqrt{3}}\left(x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2\right)$$
$$t = \frac{1}{2}\left(x_2^2 - x_3^2\right)$$. *Note that this is a basis, as* $\{r,s,t\}$ *allow for any linear combination of the* x_i^2,
$$u = x_1x_2$$
$$v = x_1x_3$$
$$w = x_2x_3$$

while $\{u,v,w\}$ *allow for any linear combination of the cross-terms. Further,* $\Psi_R(r) = r$. *Side note: the seemingly strange coefficients in front of* s *and* t *are determined so that* $\{s,t,u,v,w\}$ *have the same mean-squared value when averaged over a sphere.*

We'll now examine how the rotations act on the five-dimensional space spanned by $\{s,t,u,v,w\}$. *Since any rotation can be generated by composing rotations around the three coordinate axes, it suffices to consider the following three rotations:*

$R_1(\theta)$, *given in coordinates by* $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,

$R_2(\theta)$, *given in coordinates by* $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, *and*

$R_3(\theta)$, *given in coordinates by* $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

For each of the above, find the coordinate transformation that relates $\{s',t',u',v',w'\}$ *to* $\{s,t,u,v,w\}$.

For $R_1(\theta)$:

$$s' = \frac{1}{\sqrt{3}}\left(x_1'^2 - \frac{1}{2}x_2'^2 - \frac{1}{2}x_3'^2\right) = \frac{1}{\sqrt{3}}\left(x_1^2 - \frac{1}{2}\left(x_2\cos\theta + x_3\sin\theta\right)^2 - \frac{1}{2}\left(-x_2\sin\theta + x_3\cos\theta\right)^2\right) = \frac{1}{\sqrt{3}}\left(x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2\right) = s$$

$$t' = \frac{1}{2}\left(x_2'^2 - x_3'^2\right) = \frac{1}{2}\left(\left(x_2\cos\theta + x_3\sin\theta\right)^2 - \left(-x_2\sin\theta + x_3\cos\theta\right)^2\right) =$$
$$\frac{1}{2}\left(x_2^2\left(\cos^2\theta - \sin^2\theta\right) - x_3^2\left(\cos^2\theta - \sin^2\theta\right)^2 + 4x_2x_3\cos\theta\sin\theta\right) = \frac{1}{2}\left(x_2^2 - x_3^2\right)\cos 2\theta + \left(x_2x_3\right)\sin 2\theta$$
$$= t\cos 2\theta + w\sin 2\theta$$
$$u' = x_1'x_2' = x_1\left(x_2\cos\theta + x_3\sin\theta\right) = x_1x_2\cos\theta + x_1x_3\sin\theta = u\cos\theta + v\sin\theta$$

$$v' = x_1' x_3' = x_1(-x_2 \sin\theta + x_3 \cos\theta) = -x_1 x_2 \sin\theta + x_1 x_3 \cos\theta = -u \sin\theta + v \cos\theta$$
$$w' = x_2' x_3' = (x_2 \cos\theta + x_3 \sin\theta)(-x_2 \sin\theta + x_3 \cos\theta) =$$
$$x_2^2(-\cos\theta \sin\theta) + x_3^2(\cos\theta \sin\theta) + x_2 x_3(\cos^2\theta - \sin^2\theta) = \quad .$$
$$-\frac{1}{2}(x_2^2 - x_3^2)(2\cos\theta \sin\theta) + x_2 x_3(\cos^2\theta - \sin^2\theta) = -t \sin 2\theta + w \cos 2\theta$$

So, in the 5-d space, $R_1(\theta)$ becomes:

$$\begin{pmatrix} s' \\ t' \\ u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos 2\theta & 0 & 0 & \sin 2\theta \\ 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & -\sin 2\theta & 0 & 0 & \cos 2\theta \end{pmatrix} \begin{pmatrix} s \\ t \\ u \\ v \\ w \end{pmatrix}.$$

For $R_2(\theta)$:

$$s' = \frac{1}{\sqrt{3}}\left(x_1'^2 - \frac{1}{2}x_2'^2 - \frac{1}{2}x_3'^2\right) = \frac{1}{\sqrt{3}}\left((x_1 \cos\theta - x_3 \sin\theta)^2 - \frac{1}{2}x_2^2 - \frac{1}{2}(x_1 \sin\theta + x_3 \cos\theta)^2\right) =$$
$$\frac{1}{\sqrt{3}}\left(x_1^2\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) - \frac{1}{2}x_2^2 + x_3^2\left(\sin^2\theta - \frac{1}{2}\cos^2\theta\right) - 3x_1 x_3 \cos\theta \sin\theta\right) =$$
$$\frac{1}{\sqrt{3}}\left((x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2)\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) + \frac{1}{2}(x_2^2 + x_3^2)\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) - \frac{1}{2}x_2^2 + x_3^2\left(\sin^2\theta - \frac{1}{2}\cos^2\theta\right) - 3x_1 x_3 \cos\theta \sin\theta\right)$$
$$s\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{2}(x_2^2 + x_3^2)\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) - \frac{1}{2}x_2^2 + x_3^2\left(\sin^2\theta - \frac{1}{2}\cos^2\theta\right) - 3x_1 x_3 \cos\theta \sin\theta\right) =$$
$$s\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{2}x_2^2\left(\cos^2\theta - 1 - \frac{1}{2}\sin^2\theta\right) + \frac{1}{2}x_3^2\left(\cos^2\theta - \frac{1}{2}\sin^2\theta + 2\sin^2\theta - \cos^2\theta\right) - 3x_1 x_3 \cos\theta \sin\theta\right) =$$
$$s\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) + \frac{1}{\sqrt{3}}\left(-\frac{3}{4}x_2^2 \sin^2\theta + \frac{3}{4}x_3^2 \sin^2\theta - 3x_1 x_3 \cos\theta \sin\theta\right) =$$
$$s\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) - \frac{\sqrt{3}}{2}t \sin^2\theta - \frac{\sqrt{3}}{2}v \sin 2\theta = s\left(\frac{1}{4} + \frac{3}{4}\cos 2\theta\right) - t\frac{\sqrt{3}}{4}(1 - \cos 2\theta) - \frac{\sqrt{3}}{2}v \sin 2\theta$$

$$t' = \frac{1}{2}\left(x_2'^2 - x_3'^2\right) = \frac{1}{2}\left(x_2^2 - (x_1 \sin\theta + x_3 \cos\theta)^2\right) = \frac{1}{2}\left(x_2^2 - (x_1 \sin\theta + x_3 \cos\theta)^2\right) =$$
$$-\frac{1}{2}x_1^2 \sin^2\theta + \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2 \cos^2\theta - x_1 x_3 \cos\theta \sin\theta =$$
$$-\frac{1}{2}\left(x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2\right)\sin^2\theta + \frac{1}{2}x_2^2\left(1 - \frac{1}{2}\sin^2\theta\right) - \frac{1}{2}x_3^2\left(\cos^2\theta + \frac{1}{2}\sin^2\theta\right) - x_1 x_3 \cos\theta \sin\theta =$$
$$-s\frac{\sqrt{3}}{2}\sin^2\theta + \frac{t}{2}(1 + \cos^2\theta) - v \cos\theta \sin\theta =$$
$$-s\frac{\sqrt{3}}{4}(1 - \cos 2\theta) + \frac{t}{4}(3 + \cos 2\theta) - \frac{v}{2}\sin 2\theta$$
$$u' = x_1' x_2' = (x_1 \cos\theta - x_3 \sin\theta)x_2 = x_1 x_2 \cos\theta - x_2 x_3 \sin\theta = u \cos\theta - w \sin\theta$$

$$v' = x_1' x_3' = (x_1 \cos\theta - x_3 \sin\theta)(x_1 \sin\theta + x_3 \cos\theta) =$$
$$x_1^2 \cos\theta \sin\theta - x_3^2 \cos\theta \sin\theta + x_1 x_3 (\cos^2\theta - \sin^2\theta) = (x_1^2 - x_3^2)\cos\theta \sin\theta + x_1 x_3 \cos 2\theta$$
$$= \left(x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2\right)\cos\theta \sin\theta + \frac{1}{2}(x_2^2 - x_3^2)\cos\theta \sin\theta + x_1 x_3 \cos 2\theta$$
$$= s\frac{\sqrt{3}}{2}\sin 2\theta + t\frac{1}{2}\sin 2\theta + v\cos 2\theta$$
$$w' = x_2' x_3' = x_2 (x_1 \sin\theta + x_3 \cos\theta) = x_1 x_2 \sin\theta + x_2 x_3 \cos\theta = u\sin\theta + w\cos\theta .$$

So, in the 5-d space, $R_2(\theta)$ becomes:

$$\begin{pmatrix} s' \\ t' \\ u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+\frac{3}{4}\cos 2\theta & -\frac{\sqrt{3}}{4}(1-\cos 2\theta) & 0 & -\frac{\sqrt{3}}{2}\sin 2\theta & 0 \\ -\frac{\sqrt{3}}{4}(1-\cos 2\theta) & \frac{3}{4}+\frac{1}{4}\cos 2\theta & 0 & -\frac{1}{2}\sin 2\theta & 0 \\ 0 & 0 & \cos\theta & 0 & -\sin\theta \\ \frac{\sqrt{3}}{2}\sin 2\theta & \frac{1}{2}\sin 2\theta & 0 & \cos 2\theta & 0 \\ 0 & 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} s \\ t \\ u \\ v \\ w \end{pmatrix}.$$

Computing $R_3(\theta)$ follows the same lines as $R_2(\theta)$, with t exchanged for $-t$ and u and v exchanged. So, in the 5-d space, $R_3(\theta)$ becomes:

$$\begin{pmatrix} s' \\ t' \\ u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+\frac{3}{4}\cos 2\theta & \frac{\sqrt{3}}{4}(1-\cos 2\theta) & -\frac{\sqrt{3}}{2}\sin 2\theta & 0 & 0 \\ \frac{\sqrt{3}}{4}(1-\cos 2\theta) & \frac{3}{4}+\frac{1}{4}\cos 2\theta & \frac{1}{2}\sin 2\theta & 0 & 0 \\ \frac{\sqrt{3}}{2}\sin 2\theta & -\frac{1}{2}\sin 2\theta & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} s \\ t \\ u \\ v \\ w \end{pmatrix}.$$

One can check that all of these 5-d matrices are rotations (they have orthonormal rows and columns), and that the all have the same trace, namely $1 + 2\cos\theta + 2\cos 2\theta$.