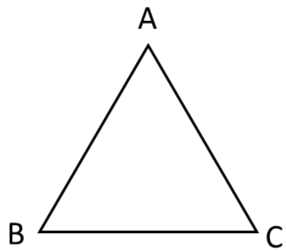
Linear Transformations and Group Representations

Homework #3 (2020-2021), Questions

Q1: Computing characters

We consider the group G of all rotations and reflections of an equilateral triangle. We will designate the group elements by the way that they permute the vertices – for example, (AB) is the reflection through vertex C and the opposite side AB that swaps vertices A and B, (ABC) is the rotation that moves A to B, B to C, and C to A. There are 6 group elements: the three pair-swaps, the two non-trivial rotations, and the identity, which we denote e.



Here we create a table of of characters several unitary representations. The first row of the table is the character of the trivial representation I, which maps every group element to 1:

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
trivial	$\chi_I(g)$:	1	1	1	1	1	1

A. Considering each element of the group as a permutation on the vertices, there is a group representation S corresponding to the sign (parity) of the permutation. What is its character?

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
sign	$\chi_S(g)$:						

B. Considering each element of the group as a rotation or reflection in the 2D plane, there is a group representation R corresponding to these 2×2 matrices. What is its character?

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
rot/ref	$\chi_R(g)$:						

C. Considering each group element as a permutation on the vertices, each group element can be represented as a 3×3 permutation matrix. This yields a representation P. What is its character?

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
permutation	$\chi_P(g)$:						

D. What is the character of the direct-sum representation $S \oplus R$?

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
sign $\oplus$ rotation	$\chi_{S\oplus R}(g)$:						

E. What is the character of the tensor-product representation $R \otimes R$?

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
rot/ref $\otimes$ rot/ref	$\chi_{R\otimes R}(g)$:						

F. Multiplication by a group element can be viewed as a permutation on the six members of the group. Writing these permutations as 6×6 permutation matrices yields the regular representation L. What is its character?

representation	g:	e	(AB)	(BC)	(AC)	(ABC)	(ACB)
regular	$\chi_L(g)$:						

Q2: Using characters

One result was that for any group representation A and B, the dimension of the space in which it acts trivially is given by $d(I,A)=\frac{1}{|G|}\sum_g \chi_A(g)$. For each of the above representations I, S, R, P, $S\oplus R$, $R\otimes R$, and L, compute $d(I,A)=\frac{1}{|G|}\sum_g \chi_A(g)$.