Linear Transformations and Group Representations

Homework #4 (2020-2021), Answers

Q1: Using characters
Our main result was that for any two group representations A and B, the number of ways that an irreducible subspace of A could be matched up with an irreducible subspace of B is given by
$d(A,B)=\frac{1}{|G|}\sum_g \overline{\chi_A(g)}\chi_B(g)$*, and that $d(A,A)=1$ -- which we here call the norm of A -- is equivalent to the statement that A is irreducible.*

A. For each of the representations I, S, R, P, $S\oplus R$, $R\otimes R$, and L in Homework 3, compute their norm. For $R\otimes R$, what irreducible representations does it contain?

We shorten the work a bit by making use of the fact that characters are constant on the pair-swaps and on the three-cycles.

representation	g:	e	$pair-swaps(3)$	$three-cycles(2)$
trivial	$\chi_I(g)$:	1	1	1
sign	$\chi_S(g)$:	1	-1	1
rot/ref	$\chi_R(g)$:	2	0	-1
permutation	$\chi_P(g)$:	3	1	0
sign $\oplus$ rot/ref	$\chi_{S\oplus R}(g)$:	3	-1	0
rot/ref $\otimes$ rot/ref	$\chi_{R\otimes R}(g)$:	4	0	1
regular	$\chi_L(g)$:	6	0	0

$$d(I,I)=\frac{1}{6}(1^2+3\bullet 1^2+2\bullet 1^2)=1$$
$$d(S,S)=\frac{1}{6}(1^2+3\bullet(-1)^2+2\bullet 1^2)=1$$
$$d(R,R)=\frac{1}{6}(2^2+3\bullet 0^2+2\bullet(-1)^2)=1$$

$$d(P,P)=\frac{1}{6}(3^2+3\bullet 1^2+2\bullet 0^2)=2$$
$$d(S\oplus R,S\oplus R)=\frac{1}{6}(3^2+3\bullet(-1)^2+2\bullet 0^2)=2$$
$$d(R\otimes R,R\otimes R)=\frac{1}{6}(4^2+3\bullet 0^2+2\bullet 1^2)=3$$
$$d(L,L)=\frac{1}{6}(6^2+3\bullet 0^2+2\bullet 0^2)=6$$

B. For $R\otimes R$, what irreducible representations does it contain?

Since the norm of $R \otimes R$ is 3, we seek three distinct irreducible representations. The above table identifies three of them, I, S, R. There cannot be any others, since there are only three conjugate classes (and the irreducible characters must be orthonormal functions on the conjugate classes). So it must be that

$R \otimes R = I \oplus S \oplus R$. This can be verified by adding the characters in the first three rows of the above table, or, by computing $d(I, R \otimes R) = d(S, R \otimes R) = d(R, R \otimes R) = 1$. For example,

$$d(R, R \otimes R) = \frac{1}{6}\left(2 \bullet 4 + 3 \bullet (0 \bullet 0) + 2 \bullet \left((-1) \bullet 1\right)\right) = 1.$$

Q2. Show that if a group is presented as a permutation of $m \geq 2$ objects, then the group representation consisting of the permutation matrices is not irreducible.

Let's call this representation M. We use the main result to show that M contains the trivial representation I by calculating $d(M, I)$. The character of M at any group element is the number of objects that are not relabeled by the permutation. So all $\chi_M(g) \geq 0$, and $\chi_M(e) = m$, since the identity element preserves the labels on all objects. The character of the trivial representation at the identity is 1 at all group elements. Therefore, the sum

$d(M, I) = \frac{1}{|G|} \sum_g \overline{\chi_M(g)} \chi_I(g)$ has at least one nonzero term ($g = e$), and all the remaining terms must be at least zero. Therefore $d(M, I) > 0$, so M must contain at least one copy of I.