Linear Transformations and Group Representations

Homework #4 (2020-2021), Questions

Q1: Using characters

Our main result was that for any two group representations $A$ and $B$, the number of ways that an irreducible subspace of $A$ could be matched up with an irreducible subspace of $B$ is given by $d(A,B)=\frac{1}{|G|}\sum_{g}\overline{\chi_A(g)}\chi_B(g)$, and that $d(A,A)=1$ -- which we here call the norm of $A$ -- is equivalent to the statement that $A$ is irreducible.

A. For each of the representations $I$, $S$, $R$, $P$, $S\oplus R$, $R\otimes R$, and $L$ in Homework 3, compute their norm. For $R\otimes R$, what irreducible representations does it contain?

We shorten the work a bit by making use of the fact that characters are constant on the pair-swaps and on the three-cycles.

| representation | $g:$ | $e$ | $pair-swaps(3)$ | $three-cycles(2)$ |
|---|---|---|---|---|
| trivial | $\chi_I(g):$ | 1 | 1 | 1 |
| sign | $\chi_S(g):$ | 1 | -1 | 1 |
| rot/ref | $\chi_R(g):$ | 2 | 0 | -1 |
| permutation | $\chi_P(g):$ | 3 | 1 | 0 |
| sign $\oplus$ rot/ref | $\chi_{S\oplus R}(g):$ | 3 | -1 | 0 |
| rot/ref $\otimes$ rot/ref | $\chi_{R\otimes R}(g):$ | 4 | 0 | 1 |
| regular | $\chi_L(g):$ | 6 | 0 | 0 |

B. For $R\otimes R$, what irreducible representations does it contain?

Q2. Show that if a group is presented as a permutation of $m\geq 2$ objects, then the group representation consisting of the permutation matrices is not irreducible.