Multivariate Methods

Homework #2 (2020-2021), Questions

Here we work out a simple multidimensional scaling problem and see how negative eigenvalues can arise. Consider four points whose distances are the entries in the following matrix:

$$D = \begin{pmatrix} 0 & 1 & b & 1 \\ 1 & 0 & 1 & b \\ b & 1 & 0 & 1 \\ 1 & b & 1 & 0 \end{pmatrix}.$$

A. Calculate the doubly-centered distance matrix G, with entries
$$G_{ij} = \frac{1}{2}\left(-d_{ij}{}^2 + \frac{1}{N}\sum_{i=1}^{N} d_{ij}{}^2 + \frac{1}{N}\sum_{j=1}^{N} d_{ij}{}^2 - \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} d_{ij}{}^2 \right).$$

B. We now find the eigenvectors of G. Observe that G, like D, is invariant under cyclic permutation of the labels (1234). Therefore, it commutes with $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, and consequently, has the same eigenvectors as P. What are the eigenvectors of P?

C. Determine the eigenvalues of G corresponding to each of the eigenvectors above.

D. Find the embedding in 3-space that corresponds to the distance matrix in A.

E. What values of b yield three equal eigenvalues?What does this indicate?

F. What values of b yield negative eigenvalues?What does this indicate?