Groups, Fields, and Vector Spaces

Homework #1 (2022-2023), Answers

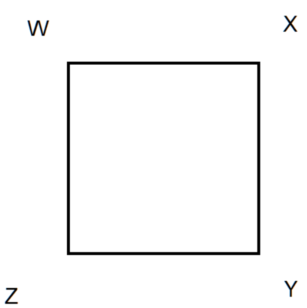
Q1: Permutations and parity: examples

Recall standard permutation notation: Every permutation can be broken down into disjoint cycles, by following around repeated application of the permutation to one element. The permutation that maps A to B, B to C, and C to A is written (ABC) (or, equivalently, (BCA) or (CAB)). The permutation that maps Q to R and R to Q is written (QR) or (RQ). The combination of the two is written, for example, $(ABC)(QR)$.

A. What is the parity of a permutation that has a single cycle of n elements?

Using the convention that a product of permutations QP means "apply P then Q", $(A_1A_2A_3...A_n) = (A_1A_2)(A_2A_3)...(A_{n-1}A_n)$. This has $n-1$ pair-swaps. So the parity of $(A_1A_2A_3...A_n)$ is $+1$ (even) if n is odd, and -1 (odd) if n is even.

B. Consider a square with vertices labeled W, X, Y, and Z in clockwise order.



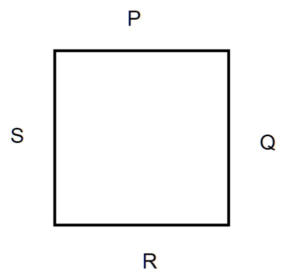
B1. Display the rotations of the square by $\pi/2$ as permutations. What is the parity?
$(WXYZ)$ and $(WZYX)$, both with odd parity (from part A with $n = 4$).

B2. Display the rotation of the square by π *as a permutation. What is the parity?*
$(WY)(XZ)$, even parity (two pair-swaps).

B3. Display the reflections that exchange one pair of opposite edges as permutations on the vertices. What is the parity?
$(WX)(YZ)$ and $(WZ)(XY)$, even parity (two pair swaps).

B4. Display the reflections that exchange one pair of opposite corners as permutations on the vertices. What is the parity?
(WY) and (XZ), odd parity (one pair swap).

C1-4. Now label the square's edges. Repeat the above, displaying the different kinds of group elements as permutations on the edges.



C1. Rotations of the square by $\pi / 2$*:*
$(PQRS)$ and $(PSQR)$, both with odd parity (from part A with $n = 4$).

C2. Rotation of the square by π*:*
$(PR)(SQ)$, even parity (two pair-swaps).

C3. Exchange one pair of opposite edges:
(SQ) and (PR), odd parity (one pair swap).

C4. Exchange one pair of opposite corners:
$(PS)(QR)$ and $(PQ)(SR)$, even parity (two pair swaps).

Q2: Permutations, parity, and homomorphisms: an application

Say G *is a finite group and* $g \in G$ *has a permutation representation with odd parity. Show that there is no element* h *in* G *for which* $g = h^2$*.*

Let φ denote the parity homomorphism between the permutation representation of G in which g has odd parity, and the group $\{1, -1\}$ (under multiplication). We are given that $\varphi(g) = -1$. If $g = h^2$, then $\varphi(g) = \varphi(h^2) = (\varphi(h))^2$; the second step is the homomorphism property. Since $\varphi(h)$ can only be 1 or -1, $(\varphi(h))^2 = 1$, which contradicts .$\varphi(g) = -1$.