Q1. A simple transfer function.

For the impulse response $f(t) = \begin{cases} \frac{1}{\tau} e^{-t/\tau}, & t \geq 0 \\ 0, & t < 0 \end{cases}$ which is the impulse response of a single-stage “RC” filter with time constant $\tau$, compute the transfer function, $\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$.

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt = \frac{1}{\tau} \int_{0}^{\infty} e^{-t/\tau} e^{-i\tau t} dt = \frac{1}{\tau} \int_{0}^{\infty} e^{-(i\omega+1/\tau)t} dt$$

$$= \frac{1}{\tau} \left. \left( e^{-(i\omega+1/\tau)t} \right) \right|_{0}^{\infty} = \frac{1}{\tau} \left( \frac{1}{1 + i\omega \tau} \right)$$

Q2. Fourier inversion via contour integration.

For the transfer function $\hat{f}(\omega)$ of Question 1, recover the Fourier transform $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega) d\omega$ via contour integration. Use a closed contour that runs along the real axis from, say, $-M$ to $M$ and then returns to its start via an excursion into either the upper- or lower- half plane.
First, take \( \tau > 0 \) and consider the integral \( I = \int_C e^{i\omega t} \hat{f}(\omega) d\omega \) over the contour in the upper half plane, illustrated below. We first show that the integrals on the segments \( I_2, I_3, \) and \( I_4 \) approach zero as \( M \to \infty \) and \( R \to \infty \). This implies that, in this limit, the contour integral is equal to \( I_1 \), which is \( 2\pi f(t) \).

To show that \( I_2 \to 0 \):

\[
|I_2| \leq \int_0^R |e^{i(M+i\tau y)}| \left| \frac{1}{1+i(M+i\tau y)} \right| dy = e^{iM} \int_0^R \frac{1}{1+i(M+i\tau)} \left| \frac{1}{1+i(M+i\tau y)} \right| dy = \int_0^R e^{-Rt} \frac{1}{1+iM-\tau^2} \left| \frac{1}{1+iM-\tau^2} \right| \frac{1}{t} = \frac{1-e^{-Rt}}{t \sqrt{(1-\tau^2)^2 + M^2\tau^2}}
\]

which, for fixed \( t \) and \( \tau \), approaches zero as \( M \to \infty \). \( I_4 \) is handled the same way, replacing \( M \) by \( -M \).

To show that \( I_3 \to 0 \):

\[
|I_3| \leq \int_M^{-M} |e^{i(x+iR\tau)}| \left| \frac{1}{1+i(x+i\tau)} \right| dx = e^{-Rt} \int_M^{-M} \frac{1}{1+i(x+i\tau)} \left| \frac{1}{1+i(x+i\tau)} \right| dx = e^{-Rt} \int_M^{-M} \frac{1}{1-R\tau+i\tau} \left| \frac{1}{1-R\tau+i\tau} \right| dx
\]

bounded away from zero, so, for fixed \( M \), \( t \) and \( \tau \), \( I_3 \) can be made as small as desired by increasing \( R \).

Evaluation of the contour integral: By Cauchy’s Theorem, the contour integral is equal to the \( 2\pi i \) times sum of the residues at all the enclosed singularities. The only singularity of

\[
\frac{1}{1+i\omega \tau} e^{i\omega t}
\]

is at \( \omega = i/\tau \), and the residue is at that point is the limiting value of the integrand

\[
\frac{1}{1+i\omega \tau} e^{i\omega t} \text{ multiplied by } \omega - i/\tau, \text{ at } \omega = i/\tau.
\]

But

\[
\frac{1}{1+i\omega \tau} e^{i\omega t} = \frac{1}{i\tau} \frac{e^{i\omega t}}{\omega - i/\tau},
\]

so the residue is

\[
\frac{1}{i\tau} e^{i(\tau/\omega)t} = \frac{1}{i\tau} e^{-i\tau/\omega}.
\]

So

\[
I = 2\pi i \left( \frac{1}{i\tau} e^{-i\tau/\omega} \right) = \frac{2\pi e^{-i\tau}}{\tau}.
\]

Since we showed that the other segments of the contour can be made arbitrarily small as \( M \to \infty \) and \( R \to \infty \), then \( I_1 \to I \) in this limit.

And in this limit, \( I_1 = 2\pi f(t) \), so

\[
f(t) = \frac{1}{\tau} e^{-i\tau}.
\]

Note that, the critical part of the argument is that \( e^{-Rt} \) can be made small by increasing \( R \). This argument holds for \( t > 0 \) since we took the contour’s return path (\( I_3 \)) to be in the upper half plane.
plane. For \( t < 0 \), the magnitude along \( I, \) can only be controlled if it is in the lower half-plane (the dashed contour in the illustration). And in the lower half-plane, \( \hat{f}(\omega) = \frac{1}{1 + i\omega \tau} \) has no singularities. So Cauchy’s Theorem says that the contour integral, and hence, \( f(t) \), is zero when \( t < 0 \).

Looking back at the above argument, we see that we didn’t need to be able to integrate \( \hat{f}(\omega) \); we just needed to know that its integral was bounded. So there’s an important bottom line: when there are no singularities of \( \hat{f}(\omega) \) in the lower half-plane and \( \hat{f}(\omega) \) behaves “nicely” for large \( \omega \), then the corresponding Fourier inverse \( f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega) d\omega \) will be the impulse response of a causal system. The converse is also true: if there are singularities of \( \hat{f}(\omega) \) in the lower half plane, then \( \hat{f}(\omega) \) cannot be the transfer function of a causal system.

Q3. When can a linear filter be realized as a continuum concatenation of another linear filter? Consider a linear filter \( L \) of a causal system, with transfer function \( \hat{L}(\omega) \).

A. If there is a linear filter \( B_2 \), for which a series combination of \( B_2 \) with itself yields \( L \), then what is \( \hat{B}_2(\omega) \)? If there is a linear filter \( B_n \), for which an series combination of \( n \) copies yields \( L \), then what is \( \hat{B}_n(\omega) \)?

We need \( (\hat{B}_2(\omega))^2 = \hat{L}(\omega) \), i.e., \( \hat{B}_2(\omega) = (\hat{L}(\omega))^{1/2} \). Similarly, \( (\hat{B}_n(\omega))^n = \hat{L}(\omega) \), i.e., \( \hat{B}_n(\omega) = (\hat{L}(\omega))^{1/n} \).

B. In the above scenario, as \( n \) grows, it seems reasonable to hypothesize that \( B_n \) becomes closer and closer to the identity – since the net result of \( n \) successive applications of \( B_n \) must remain fixed. What is \( \hat{G}(\omega) = \lim_{n \to \infty} n(\hat{B}_n(\omega) - 1) \)? If this limit exists, then \( G \) can be regarded as the infinitesimal transformation that generates \( L \), since \( \hat{B}_n(\omega) \approx 1 + \frac{1}{n} \hat{G}(\omega) \).

We want \( \lim_{n \to \infty} n((\hat{L}(\omega))^{1/n} - 1) = \lim_{\varepsilon \to 0} \frac{(\hat{L}(\omega))^\varepsilon - 1}{\varepsilon} \). Apply L’hopital’s Rule. The derivative of the denominator is 1. So we need the derivative of the numerator:

\[
\hat{G}(\omega) = \lim_{n \to \infty} n((\hat{L}(\omega))^{1/n} - 1) = \frac{d}{d \varepsilon} \left( (\hat{L}(\omega))^{\varepsilon} - 1 \right)_{\varepsilon=0} = \frac{d}{d \varepsilon} \left( e^{\varepsilon \log \hat{L}(\omega)} - 1 \right)_{\varepsilon=0}. \quad \text{Or, } \hat{L}(\omega) = e^{\hat{G}(\omega)}.
\]

\[
= \left( \log \hat{L}(\omega) e^{\varepsilon \log \hat{L}(\omega)} - 1 \right)_{\varepsilon=0} = \log \hat{L}(\omega)
\]
C. There is a converse of Q2: if there are singularities of \( \hat{f}(\omega) \) in the lower half plane, then \( \hat{f}(\omega) \) cannot be the transfer function of a causal system. So, given that \( L \) is a causal system (and therefore, that \( \hat{L}(\omega) \) has no singularities in the lower half plane), does it follow that every causal system has a causal infinitesimal? If not, what is an example?

No. Since \( \hat{G}(\omega) = \log \hat{L}(\omega) \), we need to check if the absence of singularities for \( \hat{L}(\omega) \) in the lower half plane guarantees that there are no singularities for \( \log \hat{L}(\omega) \). The logarithm has a singularity when its argument is either infinity or zero. So if \( \hat{L}(\omega) \) has a zero in the lower half plane – which is not a singularity -- then \( \hat{G}(\omega) = \log \hat{L}(\omega) \) has a singularity.

A simple example is \( \hat{h}(\omega) = \frac{1 - i\omega\tau}{1 + i\omega\tau} \), which is related to the transfer function in Q1 by \( \hat{h}(\omega) = 2\hat{f}(\omega) - 1 \). \( \hat{h}(\frac{-i}{\tau}) = 0 \), so, although \( \hat{h}(\omega) \) does not have a singularity in the lower half plane, \( \log \hat{h}(\omega) \) does.

Comment: Transfer functions that have causal infinitesimals – equivalently, that do not have zeros in the lower half plane – are called “minimum phase” transfer functions.

\( \hat{h}(\omega) = \frac{1 - i\omega\tau}{1 + i\omega\tau} \) is the archetype of a transfer function that is not minimum-phase. Note that \( |\hat{h}(\omega)| = 1 \) -- so that concatenation with this filter adds a phase shift, but without changing amplitudes.