Linear Systems: Black Boxes and Beyond

Homework #1 (2022-2023), Questions

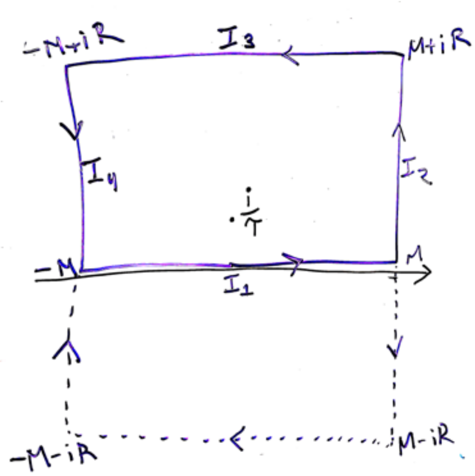
Transfer functions and complex-analytic properties.

Q1. A simple transfer function.

For the impulse response $f(t)=\begin{cases}\frac{1}{\tau}e^{-t/\tau} & ,t\geq 0\\ 0\ ,\ t<0\end{cases}$ -- which is the impulse response of a single-stage “RC” filter with time constant --, compute the transfer function, $\hat{f}(\omega)=\int\limits_{-\infty}^{\infty}e^{-i\omega t}f(t)dt$.

Q2. Fourier inversion via contour integration.

For the transfer function $\hat{f}(\omega)$ of Question 1, recover the Fourier transform $f(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{i\omega t}\hat{f}(\omega)d\omega$ via contour integration. Use a closed contour that runs along the real axis from, say, $-M$ to M and then returns to its start via an excursion into either the upper- or lower- half plane.



Q3. When can a linear filter be realized as a continuum concatenation of another linear filter? Consider a linear filter L of a causal system, with transfer function $\hat{L}(\omega)$.

A. If there is a linear filter B_2, for which a series combination of B_2 with itself yields L, then what is $\hat{B}_2(\omega)$? If there is a linear filter B_n, for which an series combination of n copies yields L, then what is $\hat{B}_n(\omega)$?

B. In the above scenario, as n grows, it seems reasonable to hypothesize that B_n becomes closer and closer to the identity – since the net result of n successive applications of B_n must remain fixed. What is $\hat{G}(\omega) = \lim_{n\to\infty} n(\hat{B}_n(\omega) - 1)$? If this limit exists, then G can be regarded as the infinitesimal transformation that generates L, since $\hat{B}_n(\omega) \approx I + \frac{1}{n}\hat{G}(\omega)$.

C. There is a converse of Q2: if there are singularities of $\hat{f}(\omega)$ in the lower half plane, then $\hat{f}(\omega)$ cannot be the transfer function of a causal system. So, given that L is a causal system (and therefore, that $\hat{L}(\omega)$ has no singularities in the lower half plane), does it follow that every causal system has a causal infinitesimal? If not, what is an example?