Transfer functions and complex-analytic properties.

Q1. A simple transfer function.
For the impulse response \( f(t) = \begin{cases} \frac{1}{\tau} e^{-t/\tau}, & t \geq 0 \\ 0, & t < 0 \end{cases} \) -- which is the impulse response of a single-stage “RC” filter with time constant --, compute the transfer function, \( \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t) dt \).

Q2. Fourier inversion via contour integration.
For the transfer function \( \hat{f}(\omega) \) of Question 1, recover the Fourier transform
\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \hat{f}(\omega) d\omega \]
via contour integration. Use a closed contour that runs along the real axis from, say, \(-M\) to \(M\) and then returns to its start via an excursion into either the upper- or lower- half plane.

Q3. When can a linear filter be realized as a continuum concatenation of another linear filter?
Consider a linear filter \( L \) of a causal system, with transfer function \( \hat{L}(\omega) \).

A. If there is a linear filter \( B_2 \), for which a series combination of \( B_2 \) with itself yields \( L \), then what is \( \hat{B}_2(\omega) \)? If there is a linear filter \( B_n \), for which an series combination of \( n \) copies yields \( L \), then what is \( \hat{B}_n(\omega) \)?
B. In the above scenario, as \( n \) grows, it seems reasonable to hypothesize that \( B_n \) becomes closer and closer to the identity – since the net result of \( n \) successive applications of \( B_n \) must remain fixed. What is \( \hat{G}(\omega) = \lim_{n \to \infty} n(\hat{B}_n(\omega) - 1) \)? If this limit exists, then \( G \) can be regarded as the infinitesimal transformation that generates \( L \), since \( \hat{B}_n(\omega) \approx I + \frac{1}{n} \hat{G}(\omega) \).

C. There is a converse of Q2: if there are singularities of \( \hat{f}(\omega) \) in the lower half plane, then \( \hat{f}(\omega) \) cannot be the transfer function of a causal system. So, given that \( L \) is a causal system (and therefore, that \( \hat{L}(\omega) \) has no singularities in the lower half plane), does it follow that every causal system has a causal infinitesimal? If not, what is an example?