Linear Systems: Black Boxes and Beyond

Homework #2 (2022-2023), Questions

Spectral Leakage

Q1. As mentioned, the amount of spectral leakage associated with a given window function $W(t)$ can be characterized by $|\hat{W}(\Delta \omega)|^2$, where $\Delta \omega = \omega - \omega_0$, $\omega_0$ is the frequency of an infinitesimally narrow spectral peak, and $\omega$ is the center of a bin of the estimated power spectrum. Here we determine the behavior of $|\hat{W}(\Delta \omega)|^2$ for some simple and popular window functions.

A. For the “square” window $W_{square}(t) = \begin{cases} 1, & |t| \leq \frac{L}{2} \\ 0, & |t| > \frac{L}{2} \end{cases}$, determine $|\hat{W}_{square}(\Delta \omega)|^2$, its behavior for large $|\Delta \omega|$, and its zeroes.

B. As in A, but for the “triangle” window $W_{triangle}(t) = \begin{cases} -\frac{1}{L} |t|, & |t| \leq \frac{L}{2} \\ 0, & |t| > \frac{L}{2} \end{cases}$.

C. As in A, but for the “cosine bell” window $W_{cosbell}(t) = \begin{cases} \frac{1}{2} \left( 1 + \cos \left( \frac{2\pi t}{L} \right) \right), & |t| \leq \frac{L}{2} \\ 0, & |t| > \frac{L}{2} \end{cases}$.

D. Plot the windows and their corresponding spectral leakage.

Q2. Algebraic properties of time- and frequency-domain restriction

Consider the vector space of square-integrable functions of time (our standard Hilbert space), and the standard inner product, $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)d\omega$ (the last equality from Parseval’s Theorem). Now consider a set of times $S_{time}$ and an arbitrary domain of (real-valued) frequencies $S_{freq}$.

Define two linear operators: $D$, defined by $Df(x) = \begin{cases} f(x), & x \in S_{time} \\ 0, & x \notin S_{time} \end{cases}$, and $B$, defined by its action on the Fourier transform of $f$: $Bf(\omega) = \begin{cases} \hat{f}(\omega), & \omega \in S_{freq} \\ 0, & \omega \notin S_{freq} \end{cases}$. In the standard development
of multitaper analysis, $S_{\text{time}}$ is an interval, and $S_{\text{freq}}$ is a range such as $|\omega| \leq \omega_{\text{max}}$; here we are dispensing with this requirement and just focusing on the algebraic properties.

A. Show that $D$ and $B$ are self-adjoint.

B. Show that that $D$ and $B$ are projections.

C. Do $D$ and $B$ commute?

D. Show that $DBD$ and $BDB$ are self-adjoint.

E. From D, we see that $DBD$ and $BDB$ are “normal” operators (they commute with their adjoints), and therefore, via the spectral theorem, their eigenvectors span the entire vector space. Show that eigenvalues of $DBD$ are also eigenvalues of $DB$, and that if $f$ is an eigenvector of $DBD$, then $Df$ is an eigenvector of $DB$, with the same eigenvalue. Similarly, if $f$ is an eigenvector of $BDB$, then $Bf$ is an eigenvector of $BD$, with the same eigenvalue.

F. Show that, for any $f$ in the vector space, that $\langle Df, Df \rangle \leq \langle f, f \rangle$ and similarly $\langle Bf, Bf \rangle \leq \langle f, f \rangle$.

G. Using F, show that all eigenvalues of $DB$ (and $BD$) are $\leq 1$. 