Linear Transformations and Group Representations

Homework #1 (2022-2023), Answers

Characteristic equations, etc.

Q1. Find the characteristic equation of $R=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$. *Find its roots, i.e., the eigenvalues of* R.

The characteristic equation is

$$\det(zI-R)=\det\begin{pmatrix}z-\cos\theta & -\sin\theta\\ \sin\theta & z-\cos\theta\end{pmatrix}=(z-\cos\theta)^2+\sin^2\theta .$$

$$=z^2-2z\cos\theta+1$$

Via the quadratic formula,

$$z=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}=\cos\theta\pm\sqrt{-\sin^2\theta}=\cos\theta\pm i\sin\theta=e^{\pm i\theta}$$

Q2. Say A *is a linear transformation on* V*, with a full set of distinct eigenvalues* $\lambda_1,\ldots,\lambda_m$*, and corresponding eigenvectors* $v_1,\ldots,v_m$*, and* B *is a linear transformation on* W*, with a full set of distinct eigenvalues* $\mu_1,\ldots,\mu_n$*, and eigenvectors* $w_1,\ldots,w_n$*. We define* $A\otimes B$ *as a linear transformation in* $V\otimes W$ *by its action on elementary tensor products* $(A\otimes B)(v\otimes w)=(Av)\otimes(Bw)$*, extended by linearity to all of* $V\otimes W$.

A. What are the eigenvalues and eigenvectors of $A\otimes B$*?*

We can build mn distinct eigenvectors from elementary tensor products of the eigenvectors in V and W, and, since mn is the dimension of $V\otimes W$, this is all of them:

$$(A\otimes B)(v_i\otimes w_j)=(Av_i)\otimes(Bw_j)=(\lambda_i v_i)\otimes(\mu_j w_j)=(\lambda_i\mu_j)(v_i\otimes w_j)$$

B. What is $tr(A\otimes B)$*, in terms of* $tr(A)$ *and* $tr(B)$*?*

Since the trace is the sum of the eigenvalues:

$$tr(A\otimes B)=\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_i\mu_j=\left(\sum_{i=1}^{m}\lambda_i\right)\left(\sum_{j=1}^{n}\mu_j\right)=(trA)(trB)$$

C. Let $A=B$ *and* $V=W$*. What are the eigenvectors and eigenvalues of* $sym(A^{\otimes 2})$*, i.e., the action of* A *in* $sym(V^{\otimes 2})$*? What are the eigenvectors and eigenvalues of* $anti(A^{\otimes 2})$*?*

For $sym(A^{\otimes 2})$: Two kinds of eigenvectors: first, the $\frac{1}{2}n(n-1)$ eigenvectors for each distinct pair of eigenvectors of A: $sym(v_i \otimes v_j) = \frac{1}{2}\left(v_i \otimes v_j + v_j \otimes v_i\right)$, and these have eigenvalues $\lambda_i \lambda_j$ (for example from part A), applied to each term. Then, there are n eigenvectors of the form $sym(v_i \otimes v_i) = v_i \otimes v_i$, and these, similarly, have eigenvalue λ_i^2.
For $anti(A^{\otimes 2})$, the $\frac{1}{2}n(n-1)$ eigenvectors are $anti(v_i \otimes v_j) = \frac{1}{2}\left(v_i \otimes v_j - v_j \otimes v_i\right)$, which have eigenvalues $\lambda_i \lambda_j$.

D. What is $tr(sym(A^{\otimes 2}))$ and $tr(anti(A^{\otimes 2}))$ in terms of $tr(A)$ and $tr(A^2)$?

Adding up the eigenvalues in C for *sym* yields $tr(sym(A^{\otimes 2})) = \sum_{i<j} \lambda_i \lambda_j + \sum_i \lambda_i^2$ and for *anti* yields $tr(anti(A^{\otimes 2})) = \sum_{i<j} \lambda_i \lambda_j$.

In general one can rearrange a sum over pairs as follows:
$\sum_{i<j} x_i x_j = \frac{1}{2}\sum_{all\, i,j} x_i x_j - \frac{1}{2}\sum_i x_i^2 = \frac{1}{2}\left(\sum_i x_i\right)^2 - \frac{1}{2}\sum_i x_i^2$. So

$tr(sym(A^{\otimes 2})) = \sum_{i<j} \lambda_i \lambda_j + \sum_i \lambda_i^2 = \frac{1}{2}\left(\sum_i \lambda_i\right)^2 + \frac{1}{2}\sum_i \lambda_i^2 = \frac{1}{2}(trA)^2 + \frac{1}{2}\sum_i \lambda_i^2$ and
$tr(anti(A^{\otimes 2})) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left(\sum_i \lambda_i\right)^2 - \frac{1}{2}\sum_i \lambda_i^2 = \frac{1}{2}(trA)^2 - \frac{1}{2}\sum_i \lambda_i^2$

To compute $tr(A^2)$: note that A^2 has eigenvalues λ_i^2 (with the same eigenvectors as A), $\sum_i \lambda_i^2 = tr(A^2)$.
So $tr(sym(A^{\otimes 2})) = \frac{1}{2}(trA)^2 + \frac{1}{2}tr(A^2)$ and $tr(anti(A^{\otimes 2})) = \frac{1}{2}(trA)^2 - \frac{1}{2}\sum_i tr(A^2)$.