Linear Transformations and Group Representations

Homework #1 (2022-2023), Answers

Characteristic equations, etc.

**Q1.** Find the characteristic equation of \( R = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \). Find its roots, i.e., the eigenvalues of \( R \).

The characteristic equation is
\[
\det (zI - R) = \det \left( \begin{array}{cc} z - \cos \theta & -\sin \theta \\ \sin \theta & z - \cos \theta \end{array} \right) = (z - \cos \theta)^2 + \sin^2 \theta.
\]
\[
= z^2 - 2z \cos \theta + 1
\]
Via the quadratic formula,
\[
z = \frac{2\cos \theta \pm \sqrt{4\cos^2 \theta - 4}}{2} = \cos \theta \pm \sqrt{-\sin^2 \theta} = \cos \theta \pm i \sin \theta = e^{\pm i\theta}
\]

**Q2.** Say \( A \) is a linear transformation on \( V \), with a full set of distinct eigenvalues \( \lambda_1, \ldots, \lambda_m \), and corresponding eigenvectors \( v_1, \ldots, v_m \), and \( B \) is a linear transformation on \( W \), with a full set of distinct eigenvalues \( \mu_1, \ldots, \mu_n \), and eigenvectors \( w_1, \ldots, w_n \). We define \( A \otimes B \) as a linear transformation in \( V \otimes W \) by its action on elementary tensor products \((A \otimes B)(v \otimes w) = (Av) \otimes (Bw)\), extended by linearity to all of \( V \otimes W \).

**A.** What are the eigenvalues and eigenvectors of \( A \otimes B \) ?
We can build \( mn \) distinct eigenvectors from elementary tensor products of the eigenvectors in \( V \) and \( W \), and, since \( mn \) is the dimension of \( V \otimes W \), this is all of them:
\[(A \otimes B)(v_i \otimes w_j) = (Av_i) \otimes (Bw_j) = (\lambda_i v_i) \otimes (\mu_j w_j) = (\lambda_i \mu_j)(v_i \otimes w_j)\]

**B.** What is \( \text{tr}(A \otimes B) \), in terms of \( \text{tr}(A) \) and \( \text{tr}(B) \) ?

Since the trace is the sum of the eigenvalues:
\[
\text{tr}(A \otimes B) = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j = \left( \sum_{i=1}^{m} \lambda_i \right) \left( \sum_{j=1}^{n} \mu_j \right) = (\text{tr}A)(\text{tr}B)
\]

**C.** Let \( A = B \) and \( V = W \). What are the eigenvectors and eigenvalues of \( \text{sym}(A^{\otimes 2}) \), i.e., the action of \( A \) in \( \text{sym}(V^{\otimes 2}) \)? What are the eigenvectors and eigenvalues of \( \text{anti}(A^{\otimes 2}) \)?
For \(\text{sym}(A^\otimes 2)\): Two kinds of eigenvectors: first, the \(\frac{1}{2}n(n-1)\) eigenvectors for each distinct pair of eigenvectors of \(A\): \(\text{sym}(v_i \otimes v_j) = \frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i)\), and these have eigenvalues \(\lambda_i \lambda_j\) (for example from part A), applied to each term. Then, there are \(n\) eigenvectors of the form \(\text{sym}(v_i \otimes v_j) = v_i \otimes v_j\), and these, similarly, have eigenvalue \(\lambda^2\).

For \(\text{anti}(A^\otimes 2)\), the \(\frac{1}{2}n(n-1)\) eigenvectors are \(\text{anti}(v_i \otimes v_j) = \frac{1}{2}(v_i \otimes v_j - v_j \otimes v_i)\), which have eigenvalues \(\lambda_i \lambda_j\).

\[\text{D. What is } \text{tr}(\text{sym}(A^\otimes 2)) \text{ and } \text{tr}(\text{anti}(A^\otimes 2)) \text{ in terms of tr}(A) \text{ and tr}(A^2)\]?

Adding up the eigenvalues in C for \(\text{sym}\) yields \(\text{tr}(\text{sym}(A^\otimes 2)) = \sum_{i \neq j} \lambda_i \lambda_j + \sum_i \lambda_i^2\) and for \(\text{anti}\) yields \(\text{tr}(\text{anti}(A^\otimes 2)) = \sum_{i < j} \lambda_i \lambda_j\).

In general one can rearrange a sum over pairs as follows:

\[
\sum_{i < j} x_i x_j = \frac{1}{2} \sum_{i \neq j} x_i x_j - \frac{1}{2} \sum_i x_i^2 = \frac{1}{2} \left( \sum_i x_i \right)^2 - \frac{1}{2} \sum_i x_i^2.
\]

So

\[
\text{tr}(\text{sym}(A^\otimes 2)) = \sum_{i < j} \lambda_i \lambda_j + \sum_i \lambda_i^2 = \frac{1}{2} \left( \sum_i \lambda_i \right)^2 + \frac{1}{2} \sum_i \lambda_i^2 - \frac{1}{2} \sum_i \lambda_i^2 + \frac{1}{2} (\text{tr}(A))^2 + \frac{1}{2} \sum \lambda_i^2 \text{ and}
\]

\[
\text{tr}(\text{anti}(A^\otimes 2)) = \sum_{i < j} \lambda_i \lambda_j = \frac{1}{2} \left( \sum \lambda_i \right)^2 - \frac{1}{2} \sum \lambda_i^2 = \frac{1}{2} (\text{tr}(A))^2 - \frac{1}{2} \sum \lambda_i^2.
\]

To compute \(\text{tr}(A^2)\): note that \(A^2\) has eigenvalues \(\lambda_i^2\) (with the same eigenvectors as \(A\)), \(\sum_i \lambda_i^2 = \text{tr}(A^2)\).

So \(\text{tr}(\text{sym}(A^\otimes 2)) = \frac{1}{2} (\text{tr}(A))^2 + \frac{1}{2} \text{tr}(A^2)\) and \(\text{tr}(\text{anti}(A^\otimes 2)) = \frac{1}{2} (\text{tr}(A))^2 - \frac{1}{2} \sum \text{tr}(A^2)\).